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On Pfaffian Random Point Fields

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Abstract We study Pfaffian random point fields by using the Moore-Dyson quaternion determinants. First, we give sufficient conditions that ensure that a self-dual quaternion kernel defines a valid random point field, and then we prove a CLT for Pfaffian point fields. The proofs are based on a new quaternion extension of the Cauchy-Binet determinantal identity. In addition, we derive the Fredholm determinantal formulas for the Pfaffian point fields which use the quaternion determinant.

Keywords Determinantal field · Determinantal point process · Random point field · Pfaffian point field · Quaternion determinant · Cauchy-Binet identity · Determinantal identity · Gaussian symplectic ensemble · Random matrices

1 Introduction

A determinantal random point field is a random collection of points with the probability distribution that can be written as a determinant. The determinantal point fields describe various mathematical objects including eigenvalues of random matrices, zeros of random analytic functions, non-intersecting random paths, and spanning trees on networks. It is conjectured that they are also related to other important objects such as Riemann's zeta function zeros and the spectrum of chaotic dynamical systems. See [10, 11, 20] and [3] for reviews.

The definition of the determinantal point field uses the standard matrix determinant. However, in some applications, the distribution of a random point field can be represented as a determinant of a quaternion matrix. An important example is provided by eigenvalues of orthogonal and symplectic random matrix ensembles. It is natural to study these fields as a generalization of the usual determinantal point fields and study their properties.

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While this generalization can be thought of as a quaternion determinantal point field, it is also equivalent to the *Pfaffian random point field*, which was recently studied in [18, 21], and [2]. In order to give a precise definition, we recall some preliminary notations.

A *random point field* $\mathcal{X} = (X, \mathcal{B}, \mathbb{P})$ on a measurable space Λ is a probability measure $\mathbb{P}$ on the space X of all possible countable configurations of points in Λ. It is convenient to think about this object as a collection of functions that sends every n-tuple of non-negative integers $(k_1, \ldots, k_n)$ and every n-tuple of measurable subsets of Λ, $(A_1, \ldots, A_n)$ to a non-negative number, which can be interpreted as a probability to find k_i points in the set A_i. These functions are to satisfy some consistency conditions, which we do not specify here. The reader is advised to consult paper [12] or book [5] for more detail. We will also call the process $\mathcal{X}$ a *Λ-valued random point field*.

Suppose that Λ is a space with measure μ. Let $\#(A_i)$ denote the number of points of $\mathcal{X}$ located in the set A_i.

Definition 1.1 A locally integrable function $R_k \colon \Lambda^k \to \mathbb{R}^1_+$ is called a *k-point correlation function* of a random point field $\mathcal{X} = (X, \mathcal{B}, \mathbb{P})$ with respect to the measure μ, if for any disjoint measurable subsets $A_1, \ldots, A_m$ of Λ and any non-negative integers $k_1, \ldots, k_m$, such that $\sum_{i=1}^m k_i = k$, the following formula holds:

$$\mathbb{E} \prod_{i=1}^{m} \left[\#(A_i) \cdots \left(\#(A_i) - k_i + 1 \right) \right] = \int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} R_k(x_1, \ldots, x_k) d\mu(x_1) \cdots d\mu(x_k), \qquad (1)$$

where $\mathbb{E}$ denote expectation with respect to measure $\mathbb{P}$.

Note that on the left is the expected number of ordered configurations of points such that set A_i contains k_i points. Note also that the correlation functions are defined only up to sets of measure 0.

Definition 1.2 A Λ-valued random point field $\mathcal{X}$ is called a *Pfaffian random point field* if its correlation functions can be written as quaternion determinants:

$$R_m(x_1, \ldots, x_m) = \mathrm{Det}_M \left(K(x_i, x_j) \right) \Big|_{1 \le i, j \le m}, \quad x_1, \ldots, x_m \in \Lambda, \ m = 1, 2, \ldots, \qquad (2)$$

where $K(x, y)$ is a self-dual quaternion kernel (that is, $K(y, x) = (K(x, y))^*$), and Det_M is the Moore-Dyson quaternion determinant.

In this definition the function $K(x, y)$ takes value in the algebra of complexified quaternions $\mathbb{Q}_{\mathbb{C}}$.

(Recall that real quaternions can be written $q = s + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where s, x, y, and z are real and where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the quaternion units with the rules $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$ and so on. The complexified quaternions are allowed to have complex coefficients s, x, y, and z. For both real and complexified quaternions, the conjugate of q is defined as $q^* = s - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. In this paper when we say quaternions, we mean complexified quaternions, and we say real quaternions for quaternions with real coefficients. Quaternion matrices are matrices whose entries are quaternions. The *dual* of a quaternion matrix X is defined as a matrix X^*, for which $(X^*)_{lk} = (X_{kl})^*$. *Self-dual* quaternion matrices are defined by the property that $X^* = X$.)

The name Pfaffian comes from a different definition of this object given in [21] and [2]. They call a random point field Pfaffian if there exists a 2×2 matrix-valued skew-symmetric kernel K on X such that the correlation functions of the process have the form

$$R_m(x_1, \dots, x_m) = \mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^m, \quad x_1, \dots, x_m \in X,\ m = 1, 2, \dots$$

(The notation Pf in the right-hand side stands for the Pfaffian.) This definition is equivalent to ours. Indeed, one can take the complex matrix representation of the quaternion kernel $K(x_i, x_j)$ and write the quaternion determinant in Definition 1.2 as a Pfaffian of a 2-by-2 matrix-valued skew-symmetric kernel equal to this representation multiplied by a matrix $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. (See formula (21) in Appendix A.)

In the opposite way, suppose that the field is Pfaffian with 2×2 matrix-valued kernel $K(x, y)$. Since

$$K(x, y) = \begin{pmatrix} A(x, y) & B(x, y) \\ -B(y, x) & D(x, y) \end{pmatrix}$$

is skew-symmetric, hence $A(y, x) = -A(x, y)$ and $D(y, x) = -D(x, y)$ and it is easy to check that

$$\begin{pmatrix} B(y, x) & -D(x, y) \\ A(x, y) & B(x, y) \end{pmatrix}$$

is a complex representation of a self-dual quaternion kernel $\widetilde{K}(x, y)$. By using the relation between the quaternion determinant and the Pfaffian, we conclude that the correlations can be written as quaternion determinants of the kernel $\widetilde{K}(x, y)$.

Here are some examples of Pfaffian fields.

Example 1 It was shown by Dyson [7] that the point field of eigenvalues of the circular orthogonal and circular symplectic ensembles of random matrices have correlation functions that can be written as quaternion determinants. Hence they form a Pfaffian point field that takes values in the unit circle of the complex plane. Later this result was extended by Mehta (see Chaps. 7 and 8 in [16]) to the case of Gaussian orthogonal and symplectic ensembles. In this case, the eigenvalues form a real-valued Pfaffian point field. We will say more about the symplectic random matrix ensembles in the last section.

Example 2 (Ginibre on $\mathbb{Q}$) Let $\Lambda = \mathbb{Q}$, where $\mathbb{Q} \simeq \mathbb{R}^4$ denotes real quaternions. Take the background measure $d\mu(z) = \pi^{-2} e^{-|z|^2} dm(z)$, where $dm(z)$ is the Lebesgue measure on $\mathbb{Q}$, and define the kernel

$$K_n(z, w) = \sum_{k=0}^{n} \frac{z^k (w^*)^k}{(k+1)!}, \quad \text{where } z, w \in \mathbb{Q} \tag{3}$$

This kernel corresponds to a Pfaffian point field that takes value in real quaternions. The fact that kernel (3) defines a valid random point field follows from Proposition 3.3.

Recall for comparison that the usual Ginibre random point field is a complex-valued point field of eigenvalues of a random n-by-n complex Gaussian matrix. It is determinantal with the kernel

$$K_n(z, w) = \sum_{k=0}^{n} \frac{z^k (\overline{w})^k}{k!},$$

where z and w are in $\mathbb{C}$ and the background measure is $d\mu(z) = \pi^{-1} e^{-|z|^2} dm(z)$. (See [9], Sect. 15.1 in [16], and Sect. 4.3.7 in [10].)

Example 3 (Pfaffian Ginibre on $\mathbb{C}$) This is another generalization of the Ginibre random point field. See Sect. 15.2 in [16] for details. It is a complex-valued Pfaffian point field. Let

$$\phi_N(u,v) = \frac{1}{2\pi} \sum_{0\le i\le j\le N-1} \frac{2^j j!}{2^i i!} \frac{1}{(2j+1)!} \left(u^{2i} v^{2j+1} - v^{2i} u^{2j+1}\right),$$

and define the quaternion kernel by its complex matrix representation:

$$\varphi\big(K_N(z,w)\big) = \begin{pmatrix} \phi_N(w,\overline{z}) & \phi_N(\overline{w},\overline{z}) \\ \phi_N(z,w) & \phi_N(z,\overline{w}) \end{pmatrix}.$$

(The map $\varphi\colon \mathbb{Q}_{\mathbb{C}} \to M_2(\mathbb{C})$ is a bijection between complexified quaternions and the 2-by-2 complex matrices. Its definition is standard and given in Appendix A.) Then this kernel defines a Pfaffian point field with respect to the signed background measure $d\mu(z) = e^{-|z|^2}(z-\overline{z})dm(z)$.

Example 4 (Bergman Kernel on $\mathbb{Q}$) Let Λ be the unit disc in $\mathbb{Q}$ with the background measure $d\mu(z) = \pi^{-2} dm(z)$, where $dm(z)$ is the Lebesgue measure on $\mathbb{Q} \simeq \mathbb{R}^4$. Define the kernel

$$K_n(z,w) = \sum_{k=0}^{n} (k+2) z^k \left(w^*\right)^k, \quad \text{where } z, w \in \mathbb{Q}. \tag{4}$$

Then this kernel corresponds to a Pfaffian point field that takes value in the unit disc of real quaternions.

For comparison, the Bergman kernel on the unit disc in $\mathbb{C}$ is given by

$$K_n(z,w) = \sum_{k=0}^{n} (k+1) z^k (\overline{w})^k,$$

and corresponds to the determinantal point field of zeros of power series with i.i.d complex Gaussian coefficients. (See Sect. 15.2 in [10].)

Which self-dual quaternion kernels correspond to random point fields?

Note that a correlation function is automatically symmetric if it is defined as in (2). That is, we have

$$R_m(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) = R_m(x_1, \ldots, x_m)$$

for every permutation $\sigma \in S_m$. Hence, by the Lenard criterion ([13] and [14]), positivity is a necessary and sufficient condition that ensures that a kernel corresponds to a random point field. This condition can be explained as follows. Let X be the space of configurations of points in Λ. Let $\varphi = \{\varphi_k\}$ denote an arbitrary sequence of real-valued functions φ_k over Λ^k which have the property that they are zero if at least one of its arguments is outside of a certain compact set in Λ. Define operator S on φ as follows: S maps sequence φ to a function $S\varphi$ on X

$$(S\varphi)(x) = \sum \varphi_k(x_{i_1}, \ldots, x_{i_k})$$

where $x = (x_1, x_2, \ldots)$ is an enumeration of a point configuration in X, and the sum is extended over all finite sequences of distinct positive integers $(i_1, \ldots, i_k)$ including the empty sequence. (This sum is essentially finite since the definition of the space of configurations requires that the number of points of any configuration in any compact subset of Λ be finite.) The positivity condition says that if $(S\varphi)(x) \geq 0$ for all $x \in X$ (including the empty sequence), then it must be true that

$$\int \varphi(x) d\rho := \varphi_0 + \sum_{k=1}^{N} \int_{\Lambda^k} \varphi_k(x_1, \ldots, x_k) R_k(x_1, \ldots, x_k) d\mu(x_1) \cdots d\mu(x_k) \geq 0.$$

This criterion is useful for fields with small number of points. However, in general this criterion is often difficult to verify.

In the case of the usual determinantal fields with self-adjoint kernels a much simpler criterion is given by the Macchi-Soshnikov theorem (see [15] and [20]) that says that a self-adjoint kernel $K(x, y)$ defines a determinantal point field if and only if the corresponding operator K is in the trace class and all its eigenvalues are in the interval [0, 1].

Our first result is a weaker version of this theorem for Pfaffian point fields.

Suppose that a quaternion kernel K can be written as follows:

$$K(x, y) = \sum_{k=1}^{\infty} \lambda_k u_k(x) u_k^*(y), \tag{5}$$

where λ_k are scalar and $u_k(x)$ is an orthonormal system of quaternion functions:

$$\int_{\Lambda} u_k^*(x) u_l(x) d\mu(x) = \delta_{kl}.$$

(The series in (5) are assumed to be absolutely convergent almost everywhere.) We will say in this case that $K(x, y)$ has a *diagonal form* with eigenvalues λ_k. If λ_k are real and $u_k(x)$ is an orthonormal system of *real quaternion* functions, then we will say that $K(x, y)$ has a *real diagonal form.*

Theorem 1.3 *Suppose that a quaternion kernel $K(x, y)$ has a real diagonal form with eigenvalues λ_k. Assume that all $\lambda_k \in [0, 1]$, and that $\sum_{k=1}^{\infty} \lambda_k < \infty$. Then the kernel $K(x, y)$ defines a Pfaffian point field with finite expected number of points.*

The conditions of this theorem are sufficient but not necessary since there exist self-dual quaternion kernels which do not have a real diagonal form and still define a valid point field. For example, let the space Λ consist of two points and has the counting background measure. Let $K = \frac{1}{2}\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix}$, where $a = (3i\mathbf{i} - 5\mathbf{j})/4$ so that $a^2 = -1$. This matrix is self-dual with determinant zero. It defines a random point field that has exactly one point uniformly distributed on Λ. This kernel has a diagonal form but it does not have a real diagonal form. (The eigenvector is a complex quaternion vector.)

Here is another example. Let Λ consist of two points, and let $K = \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}$, where $a = i\mathbf{i} - \mathbf{j}$, so that $a^2 = 0$. This matrix is self-dual with determinant 1. It defines a random point field with exactly 2 points and correlation functions $R_1 = 1$ and $R_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. On the other hand, K does not have a diagonal form. (The eigenvalues equal 1 but $K^2 \neq K$.)

These examples shows that the conditions of Theorem 1.3 are sufficient but not necessary. On the other hand, it is not possible to drop entirely requirements on the eigenfunctions $u_k(x)$. For example, let Λ consist of two points and let $K = \frac{4}{3}\begin{pmatrix} 1 & i/2 \\ i/2 & -1/4 \end{pmatrix}$. (We consider this matrix as a quaternion matrix. It is self-dual since $(i/2)^* = i/2$.) This matrix has eigenvalues 1 and 0 and it can be computed that it has a diagonal form with $\lambda = 1$

and $u = (2/\sqrt{3})[1, i/2]^*$. However, it does not define a valid random field since the first correlation function is negative at the second point.

Many of the examples from random matrix theory are concerned with kernels that do not have a real diagonal form. For example, one can check that the circular orthogonal ensemble corresponds to a kernel without a diagonal form and the circular symplectic ensemble corresponds to a kernel without a real diagonal form. Hence, it appears desirable to find an extension of Theorem 1.3 . Ideally, we would like to know the sufficient and necessary conditions that would ensure that a self-dual quaternion kernel defines a valid random point field.

While this question is not answered in this paper, we can extend Theorem 1.3 and give sufficient conditions that ensure that a kernel without a real diagonal form defines a valid random point field. First, let us say that a kernel $K(x, y)$ has a *quasi-real diagonal form* (or simply call it *quasi-real*) if it has a diagonal form with real eigenvalues. Second, we will call it *positive* if

$$\mathrm{Det}_M\big(K(x_i, x_j)\big)\big|_{1\le i,j\le m} \ge 0$$

for all $m \in \mathbb{Z}^+$ and all $x_1, \ldots, x_m \in \Lambda$. (A remarkable fact is that quasi-real kernels with positive eigenvalues are not necessarily positive, as the last example above shows.)

We will call a quasi-real kernel $K(x, y)$ *completely positive* if every of the kernels

$$K_I = \sum_{i\in I} u_i(x) u_i^*(y)$$

is positive, where $\{u_i(x)\}$ is an orthonormal set of eigenfunctions of $K(x, y)$ and $I = (i_1, \ldots, i_m)$ denote an ordered subset of indices of all eigenfunctions.

Theorem 1.4 *Suppose that a quaternion kernel $K(x, y)$ is finite-rank, quasi-real and completely positive. Assume that all $\lambda_k \in [0, 1]$. Then the kernel $K(x, y)$ defines a Pfaffian point field.*

Remark: We omit the assumption that $\sum_{k=1}^{\infty} \lambda_k < \infty$ which we imposed in Theorem 1.3 since we assume that the kernel is finite-rank. It should be possible to extend this theorem to a more general case when there are infinite number of λ_k and $\sum_{k=1}^{\infty} \lambda_k < \infty$.

Still, even if we restrict attention to kernels with a diagonal form, the conditions of Theorem 1.4 are not necessary. For example, consider the two-point space Λ with the counting measure μ. Let $a = (1 + 2i) + (\frac{19}{10} - \frac{20}{19} i)\mathbf{i}$, and define the kernel as $K = \frac{1}{2}\left(\begin{smallmatrix} 1 & a \\ a^* & 1 \end{smallmatrix}\right)$. Then $\mathrm{Det}_M(K) \approx 0.3745$, the pair correlation function is positive, and the kernel defines a valid random point field on Λ. On the other hand the eigenvalues are $\lambda_{1,2} \approx \frac{1}{2} \pm 0.3529i$ and therefore the kernel is not quasi-real.

In order to prove Theorems 1.3 and 1.4, let us introduce the following notations.

Suppose $K(x, y)$ has a diagonal form with eigenvalues λ_k, that all $\lambda_k \in [0, 1]$, and that $\sum_{k=1}^{\infty} \lambda_k < \infty$. Define a random kernel

$$K_\xi(x, y) = \sum_{k=1}^{\infty} \xi_k u_k(x) u_k^*(y), \tag{6}$$

where ξ_k are independent Bernoulli random variables. The random variable ξ_k takes value 1 with probability λ_k. (We will prove later that this kernel defines a valid point field.)

Theorems 1.3 and 1.4 are immediate consequences of the following results.

Theorem 1.5 *Suppose $K(x, y)$ has a real diagonal form with eigenvalues λ_k, that all $\lambda_k \in [0, 1]$, and that $\sum_{k=1}^{\infty} \lambda_k < \infty$. Let $\mathcal{X}_\xi$ be a random point field which is a mix of Pfaffian point fields with random kernels $K_\xi(x, y)$ defined in (6). Then, $\mathcal{X}_\xi$ is a Pfaffian point field with the kernel $K(x, y)$.*

An analogous result holds for the quasi-real case.

Theorem 1.6 *Suppose $K(x, y)$ is finite-rank, quasi-real and completely positive with eigenvalues $\lambda_k \in [0, 1]$. Let $\mathcal{X}_\xi$ be a random point field which is a mix of Pfaffian point fields with random kernels $K_\xi(x, y)$ defined in (6). Then, $\mathcal{X}_\xi$ is a Pfaffian point field with the kernel $K(x, y)$.*

These theorems are quaternion analogues of Theorem 7 in [11]. The key ingredient in their proofs is an analogue of the Cauchy-Binet formula for quaternion determinants which we state in Sect. 2 and prove in Appendix B. We will prove these theorems in Sect. 3.

New kernels of Pfaffian point fields can also be obtained by the restriction operation.

Proposition 1.7 *Suppose $K(x, y)$ is a kernel of a Λ-valued Pfaffian point field $\mathcal{X}$. Let $D \subset \Lambda$. Then $K_D(x, y) = \mathbf{1}_D(x) K(x, y) \mathbf{1}_D(y)$ is a kernel of another Λ-valued Pfaffian point field.*

Proof The correlation functions defined by the kernel $K_D(x, y)$ are valid correlation functions since they equal the correlations functions of a random point field $\mathcal{X}_D$, generated by the following procedure. First, generate the points of $\mathcal{X}$. Then remove the points which are outside of D. □

In addition to the existence results, Theorem 1.5 allows us to study the total number of points in a Pfaffian point field.

Theorem 1.8 *Let $\mathcal{X}$ be a Λ-valued Pfaffian point field with a finite-rank kernel K. Let $\mathcal{N}$ denote the number of points of $\mathcal{X}$ in Λ. Then the characteristic function of the random variable $\mathcal{N}$ satisfies the following equations*:

$$\begin{aligned} \varphi_{\mathcal{N}}(t) \equiv \mathbb{E}\big(e^{i\mathcal{N}t}\big) &= \prod_{k=1}^{r} \big(1 + \big(e^{it} - 1\big)\lambda_k\big) \\ &= \mathrm{Det}_M\big(I + \big(e^{it} - 1\big)K\big), \end{aligned} \tag{7}$$

where λ_k are eigenvalues of the kernel K and r is its rank.

A consequence of this result is the central limit theorem for the number of points in Pfaffian point fields.

Theorem 1.9 *Let $\mathcal{X}_n$ be a sequence of Λ_n-valued Pfaffian point fields with finite-rank kernels K_n. Let $\mathcal{N}_n$ denote the number of points of $\mathcal{X}_n$. Suppose that all eigenvalues of kernels K_n are real and in the interval $[0, 1]$, and that $\mathbb{V}ar(\mathcal{N}_n) \to \infty$ as $n \to \infty$. Then, the sequence of random variables $(\mathcal{N}_n - \mathbb{E}(\mathcal{N}_n))/\sqrt{\mathbb{V}ar(\mathcal{N}_n)}$ approaches a standard Gaussian random variable in distribution.*

This is an analog of a theorem that was proved by Costin and Lebowitz in [4] and Diaconis and Evans in [6] for particular cases, and by Soshnikov in [22] for general determinantal ensembles. Later, a simplified proof was suggested in [11] and we use its main idea to prove our theorem.

We will see in the final section that the circular and Gaussian symplectic ensemble of random matrices have finite-rank projection kernels . Numerical evaluations suggest that the restrictions of these kernels have real eigenvalues in the interval [0, 1]. However, the proof of this claim is elusive.

In the theory of determinantal random fields, a prominent place is given to determinantal formulas for evaluation of expressions like $\mathbb{E}\prod_{k=1}^{N}(1+f(x_k))$, where x_k denote the points of the field. These formulas are useful for calculating the distribution of spacings and similar quantities. It turns out that similar expressions can be written for Pfaffian fields. (This was also observed in Rains [18] in a somewhat different form.)

First, we define the quaternion version of the Fredholm determinant for the self-dual kernel $K(x, y)$:

$$\mathrm{Det}_M(I+K) := 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\mathrm{Det}_M\big(K(x_i,x_j)\big)\big|_{1\le i,j\le n}dx_1\cdots dx_n.$$

Note that for finite-rank kernels the series has only finite number of terms. In addition, it can be checked that if Λ is finite, so that $I+K$ is a matrix, then this definition agrees with the usual definition of the Dyson-Moore determinant for the matrix $I+K$.

Let us consider $\mathbb{E}\prod_k(1+f(x_k))$, where $f(x)$ is a complex-valued function and the product extended over all points of the field.

Theorem 1.10 *Suppose that the kernel K of a Pfaffian field has finite rank. Then*

$$\mathbb{E}\prod_k\big(1+f(x_k)\big) = \mathrm{Det}_M\big(I+\sqrt{f(x)}K\sqrt{f(y)}\big),$$

whenever terms on both sides of this equality are well-defined.

Proof The assumption implies that the field has a finite number of points and the product $\mathbb{E}\prod_k(1+f(x_k))$ has finite number of terms. By the definition of correlation functions

$$\begin{aligned}\mathbb{E}\prod_i\big(1+f(x_i)\big) &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\left(\prod_{i=1}^{n}f(x_i)\right)R(x_1,\ldots,x_n)dx_1\cdots dx_n\\ &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\left(\prod_{i=1}^{n}f(x_i)\right)\mathrm{Det}_M\big(K(x_i,x_j)\big)\big|_{1\le i,j\le n}dx_1\cdots dx_n\\ &= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\mathrm{Det}_M\big(\sqrt{f(x_i)}K(x_i,x_j)\sqrt{f(x_j)}\big)\big|_{1\le i,j\le n}dx_1\cdots dx_n\\ &= \mathrm{Det}_M\big(I+\sqrt{f(x)}K\sqrt{f(y)}\big).\end{aligned}$$

(Note that in fact all the series in this calculation have the finite number of terms.) □

In particular, formula (7) can be alternatively obtained from Theorem 1.10 by taking $f(x)=e^{it}-1$, a function which is constant in x and depends only on a parameter t.

The rest of the paper is organized as follows. Section 2 formulates a quaternion version of the Cauchy-Binet identity. Section 3 proves the existence Theorems 1.3 and 1.4, Sect. 4 proves the CLT in Theorem 1.9, and Sect. 5 provides an illustration by considering the circular and Gaussian symplectic ensembles of random matrices. Appendix A contains background information about quaternion matrices and determinants and Appendix B contains a proof of the quaternion Cauchy-Binet identity.

2 Cauchy-Binet Formula

Determinants of quaternion matrices have been studied for almost a hundred years. (Study wrote a paper [23] about them in 1920, and Moore presented his definition [17] in 1922). In the 1970s, this subject got a boost after Dyson [7] re-discovered Moore's determinant and related it to the distribution of eigenvalues of random matrices.

Unfortunately, none of the available quaternion determinants enjoys all the properties of the usual determinant, and the validity of each standard determinantal identity has to be checked individually.

The Cauchy-Binet identity states that if A is an m-by-n matrix and B is an n-by-m matrix, with $n \geq m$, then

$$\det(AB) = \sum_I \det\bigl(A^I\bigr)\det\bigl(B^I\bigr),$$

where the summation is over $I = (i_1 < i_2 < \cdots < i_m)$, the ordered subsets of $\{1, \ldots, n\}$ that consist of m elements. Matrices A^I and B^I are square matrices that consist of m columns of A and m rows of B, respectively, with indices in I.

An implicit assumption in this result is that the entries of the matrices A and B are from a commutative ring, for example from a field of complex numbers. Unfortunately, in this form the Cauchy-Binet identity fails for the quaternion determinants.

However, a weaker form of the Cauchy-Binet identity still holds.

Theorem 2.1 *Let C be an n-by-m quaternion matrix, $n \geq m$, and let C^* be the dual of C. Then*

$$\mathrm{Det}_M\bigl(C^* C\bigr) = \sum_I \mathrm{Det}_M\bigl(\bigl(C^I\bigr)^* C^I\bigr),$$

where the summation is over $I = (i_1 < i_2 < \cdots < i_m)$, the ordered subsets of $\{1, \ldots, n\}$ that consist of m elements, and where C^I consists of rows $i_1, \ldots, i_m$ of C.

A proof of this theorem is in Appendix B.

Corollary 2.2 *Suppose that Λ is an n-by-n diagonal matrix with scalar entries λ_i on the main diagonal and that C is an n-by-m quaternion matrix, $n \geq m$. Then, we have.*

$$\mathrm{Det}_M\bigl(C^* \Lambda C\bigr) = \sum_{\substack{I=(i_1,\ldots,i_m)\\ i_1<\cdots<i_m}} \lambda_{i_1} \cdots \lambda_{i_m} \mathrm{Det}_M\bigl(\bigl(C^I\bigr)^* C^I\bigr).$$

Proof of Corollary 2.2 Let $\Lambda^{1/2}$ be an n-by-n diagonal matrix with scalar entries such that $(\Lambda^{1/2})^2 = \Lambda$. For $I = (i_1, \ldots, i_m)$, let $(\Lambda^{1/2})^{II}$ denote an m-by-m matrix which is formed by taking entries at the intersection of rows and columns $i_1, \dots, i_m$ in matrix $\Lambda^{1/2}$. We write

$$\begin{aligned}
\mathrm{Det}_M\big(C^*\Lambda C\big) &= \sum_I \mathrm{Det}_M\big(\big(\Lambda^{1/2}C\big)^{I*}\big(\Lambda^{1/2}C\big)^I\big) \\
&= \sum_I \mathrm{Det}_S\big(\big(\Lambda^{1/2}C\big)^I\big) \\
&= \sum_I \mathrm{Det}_S\big(\big(\Lambda^{1/2}\big)^{II}C^I\big) \\
&= \sum_I \mathrm{Det}_S\big(\big(\Lambda^{1/2}\big)^{II}\big)\,\mathrm{Det}_S\big(C^I\big) \\
&= \sum_{I=(i_1,\dots,i_m)} \lambda_{i_1}\cdots\lambda_{i_m}\,\mathrm{Det}_M\big(\big(C^I\big)^*C^I\big).
\end{aligned}$$

The first line is the Cauchy-Binet identity. The second line is the relation between the Moore-Dyson and Study determinants (see (23) in Appendix A). The fourth line follows by multiplicativity of the Study determinant. And in the fifth line we have used (23) again. □

3 The Existence of Pfaffian Point Fields

Proposition 3.1 *Let $K_N(x, y) = \sum_{k=1}^N u_k(x)u_k^*(y)$, where $u_k(x)$ are orthonormal quaternion functions, and assume that $K_N(x, y)$ is positive, that is,*

$$\mathrm{Det}_M\big(K_N(x_i, x_j)\big)\big|_{1\le i,j\le m} \ge 0$$

for all $m \in \mathbb{Z}^+$ and all $x_1, \dots, x_m$. Then $K_N(x, y)$ defines a valid symplectic determinantal field with exactly N points.

Note that for the case when $u_k(x)$ are real quaternion functions the assumption of positivity can be dropped. Indeed, in this case, the matrix $K = K_r(x_i, x_j)|_{i,j=1,\dots,m}$ is positive semidefinite. (That is, for every real quaternion vector v, $v^*Kv \ge 0$.) This implies that all eigenvalues of this matrix are real and non-negative. For self-dual matrices with real quaternion entries the Moore-Dyson determinant can be computed as the product of the eigenvalues, and we conclude that $\mathrm{Det}_M(K_N(x_i, x_j))|_{1\le i,j\le m} \ge 0$.

Proof of Proposition 3.1 We need to show that the functions defined by the rule

$$R_m(x_1, \dots, x_m) = \mathrm{Det}_M\big(K_N(x_i, x_j)\big), \quad i, j = 1, \dots, m$$

are the correlation functions of a random field.

First, by assumption of positivity all functions $R_m(x_1, \dots, x_m)$ are non-negative.

Next, by integrating the kernel we find that

$$\int_{\mathbb{R}} K_N(x, x)dx = N,$$

$$\int_{\mathbb{R}} K_N(x, y)K_N(y, x)dydx = N.$$

By using formulas (1), we can conclude that the total number of points in the process with kernel $K_N(x, y)$ is exactly N. (Its expectation is N and its variance is 0.) Hence, it remains to show that the functions $R_m(x_1, \dots, x_m)$ agree among themselves for all m. This can be done by the quaternion analogue of the Mehta lemma. (Compare Theorem 5.1.4 on p. 75 in Mehta [16])

Let $K_m := (K(x_i, x_j))_{1 \le i, j \le m}$ for a self-dual quaternion kernel $K(x, y)$. □

Lemma 3.2 (Dyson) *Assume that K satisfies either*

$$\int_{\mathbb{R}} K(x, y)K(y, z)dy = K(x, z) \tag{8}$$

or

$$\varphi\Big(\int_{\mathbb{R}} K(x, y)K(y, z)dy\Big) = \varphi\big(K(x, z)\big) + E\varphi\big(K(x, z)\big) - \varphi\big(K(x, z)\big)E, \tag{9}$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{10}$$

Then

$$\int_{\mathbb{R}} \mathrm{Det}_M(K_m)dx_m = (N - m + 1)\,\mathrm{Det}_M(K_{m-1}), \tag{11}$$

where $N = \int K(x, x)d\mu(x)$.

This result is due to Dyson. (See proof of Theorem 4 in [7].)

For the kernel $K_N(x, y)$, equation (8) holds, and we find that

$$\int_{\mathbb{R}} R_m(x_1, \dots, x_m)dx_m = \big(N - (m - 1)\big)R_{m-1}(x_1, \dots, x_{m-1}),$$

which shows that all correlation functions are all in agreement.

Corollary 3.3 *Assume that*

$$K_\xi(x, y) = \sum_{k=1}^{\infty} \xi_k u_k(x)u_k^*(y), \tag{12}$$

where every ξ_k is an independent Bernoulli random variable that takes value 1 with probability λ_k, and where $u_k(x)$ are orthonormal real quaternion functions. Suppose that $\sum_{k=1}^{\infty} \lambda_k < \infty$. Conditional on ξ, the function $K_\xi(x, y)$ is a kernel of a determinantal point field, $\mathcal{X}_\xi$, and the total number of points in $\mathcal{X}_\xi$ is $\sum_{k=1}^{\infty} \xi_k$.

We can prove an analogous result for a quasi-real diagonal form.

Corollary 3.4 *Assume that*

$$K_\xi(x, y) = \sum_{k=1}^{N} \xi_k u_k(x)u_k^*(y), \tag{13}$$

where every ξ_k is an independent Bernoulli random variable that takes value 1 *with probability λ_k. Suppose that the kernel $K(x, y) = \sum_{k=1}^{N} u_k(x)u_k^*(y)$ is completely positive. Conditional on ξ, the function $K_\xi(x, y)$ is a kernel of a determinantal point field, $\mathcal{X}_\xi$ and the total number of points in $\mathcal{X}_\xi$ is $\sum_{k=1}^{N} \xi_k$.*

Proof of Theorem 1.5 By the iterated expectation formula, the correlation functions of process $\mathcal{X}_\xi$ equal $\mathbb{E}\,\mathrm{Det}_M(K_\xi(x_i, x_j))$, where the expectation is taken over randomness in ξ. Hence, it is enough to prove that

$$\mathbb{E}\,\mathrm{Det}_M\big(K_\xi(x_i, x_j)\big) = \mathrm{Det}_M\big(K(x_i, x_j)\big), \quad i, j = 1, \dots, m, \tag{14}$$

almost everywhere.

First, let

$$K_R(x, y) = \sum_{k=1}^{R} \lambda_k u_k(x)u_k^*(y).$$

Since we assumed the absolute convergence of the kernel, hence $\mathrm{Det}_M(K_R(x_i, x_j))$. converges to $\mathrm{Det}_M(K(x_i, x_j))$ almost everywhere as $R \to \infty$.

Let R-by-m matrix C be defined as

$$C_{kl} = u_k^*(x_l), \quad k = 1, \dots, R; \quad l = 1, \dots, m.$$

and let Λ be an R-by-R diagonal matrix with diagonal entries λ_i.

By Corollary 2.2,

$$\mathrm{Det}_M\big(K_R(x_i, x_j)\big) = \sum_{\substack{I=(i_1,\dots,i_m)\\ i_1<\dots<i_m}} \lambda_{i_1}\cdots\lambda_{i_m}\,\mathrm{Det}_M\big(\big(C^I\big)^* C^I\big). \tag{15}$$

Next, let the random variable $\mathrm{Det}_M(K_\xi(x_i, x_j))$ be denoted as Y and let A_R be the event that all ξ_k are zero for $k > R$. (That is, $A_R = \bigcap_{k>R}\{\xi_k = 0\}$). Note that

$$\mathbb{E}Y = \mathbb{E}(Y|A_R)\mathbb{P}(A_R\}) + \mathbb{E}\big(Y|A_R^c\big)\mathbb{P}\big(A_R^c\big). \tag{16}$$

By using independence of A_R and ξ_k for $k \le R$, we find that

$$\mathbb{E}(Y|A_R)\mathbb{P}(A_R) = \mathbb{E}\,\mathrm{Det}_M\big(C^*\Lambda_\xi C\big)\mathbb{P}(A_R),$$

where Λ_ξ denotes an R-by-R diagonal matrix with diagonal entries ξ_i.

Next, by Corollary 2.2,

$$\mathbb{E}\,\mathrm{Det}_M\big(C^*\Lambda_\xi C\big) = \mathbb{E}\sum_{\substack{I=(i_1,\dots,i_m)\\ i_1<\dots<i_m}} \xi_{i_1}\cdots\xi_{i_m}\,\mathrm{Det}_M\big(\big(C^I\big)^* C^I\big).$$

Since the variables $\xi_{i_1}, \dots, \xi_{i_m}$ are independent and have expectation $\lambda_{i_1}, \dots, \lambda_{i_m}$, we find that $\mathbb{E}\xi_{i_1}\cdots\xi_{i_m} = \lambda_{i_1}\cdots\lambda_{i_m}$. Hence

$$\mathbb{E}(Y|A_R\})\mathbb{P}(A_R) = \mathrm{Det}_M\big(K_R(x_i, x_j)\big)\mathbb{P}(A_R), \tag{17}$$

and the probability $\mathbb{P}(A_R)$ converges to 1 as $R \to \infty$ by independence of ξ_k, Borel-Cantelli lemma and the assumption $\sum \lambda_k < \infty$.

Now let us show that $\mathbb{E}(Y|A_R^c)\mathbb{P}(A_R^c)$ converges to zero almost everywhere. By positivity of the determinant, it is enough to show that

$$\int_{\mathbb{R}^m} \mathbb{E}\big(\mathrm{Det}_M\big(K_\xi(x_i,x_j)\big)\big|A_R^c\big)\mathbb{P}\big(A_R^c\big) \to 0 \tag{18}$$

as $R \to \infty$.

Let

$$n_\xi := \sum_{k=1}^{\infty} \xi_k,$$

which is finite since both the expectation and the variance of the sum on the right hand side are convergent. Since K_ξ is a projection operator, the total number of points of the process $\mathcal{X}_\xi$ in $\mathbb{R}$ equals n_ξ. By changing the order of the expectation and the integral signs in (18), which is possible since the integrand is positive, and by using the identities for correlation functions we obtain that we need to estimate

$$\mathbb{E}\big(n_\xi(n_\xi-1)\cdots(n_\xi-m+1)|A_R^c\big)\mathbb{P}\big(A_R^c\big),$$

which is smaller than

$$\mathbb{E}\big(n_\xi^m|A_R^c\big)\mathbb{P}\big(A_R^c\big).$$

By expanding

$$n_\xi^m = \left(\sum_{k=1}^{\infty} \xi_k\right)^m,$$

and using the fact that $\xi_k^s = \xi_k$ for every integer $s \geq 1$, we observe that it is enough to show that

$$\mathbb{E}\left(\sum_{i_1<\cdots<i_r} \xi_{i_1}\xi_{i_2}\cdots\xi_{i_r}|A_R^c\right)\mathbb{P}\big(A_R^c\big) \to 0,$$

as $R \to \infty$, where the sum is over all possible ordered r-tuples $(i_1,\ldots,i_r)$ such that $i_1 < \cdots < i_r$, and $1 \leq r \leq m$.

We can divide the sum in two parts. The first part is when $i_r \leq R$. In this case, the variables $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_r}$ are independent from the event A_R^c, and therefore we can estimate this part of the sum as

$$\left(\sum_{i_1<\cdots<i_r} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}|A_R^c\right)\mathbb{P}\big(A_R^c\big) \leq cS^m\mathbb{P}\big(A_R^c\big),$$

where $S := \sum_{k=1}^{\infty} \lambda_k$ and c is an absolute constant. Hence this part converges to zero as $R \to \infty$ because $\mathbb{P}(A_R^c)$ converges to zero.

The second part of the sum is when $i_r > R$. In this case, the event $\xi_{i_r} = 1$ implies A_R^c and therefore,

$$\begin{aligned}\mathbb{E}\big(\xi_{i_1}\xi_{i_2}\cdots\xi_{i_r}|A_R^c\big)\mathbb{P}\big(A_R^c\big) &= \mathbb{P}\big(\xi_{i_1}=1,\ldots,\xi_{i_r}=1, \text{ and } A_R^c\big)\\ &= \mathbb{P}(\xi_{i_1}=1,\ldots,\xi_{i_r}=1)\\ &= \lambda_{i_1}\cdots\lambda_{i_r}.\end{aligned}$$

Therefore, we estimate the second part as

$$\sum_{\substack{i_1<\cdots<i_r,\\ i_r>R}} \lambda_{i_1}\cdots\lambda_{i_r} \le \sum_{i_r=R+1}^{\infty} \sum_{i_1,\ldots,i_{r-1}=1}^{\infty} \lambda_{i_1}\cdots\lambda_{i_r}$$
$$= S^{m-1} \sum_{i_r=R+1}^{\infty} \lambda_{i_r}.$$

Hence the second part converges to zero as $R\to\infty$, because the tail sums $\sum_{i_r=R+1}^{\infty}\lambda_{i_r}$ converge to zero.

This shows that $\mathbb{E}(Y|A_R^c)\mathbb{P}(A_R^c)$ in (16) converges to zero as $R\to\infty$. If we compare (15) and (17) and let R grow to infinity, we find that

$$\mathbb{E}\,\mathrm{Det}_M\big(K_\xi(x_i,x_j)\big) = \mathrm{Det}_M\big(K(x_i,x_j)\big) \tag{19}$$

almost everywhere, and this completes the proof of the theorem. □

The proof of Theorem 1.6 is essentially the same as for Theorem 1.5 but it is easier, since we assumed that the number of eigenvalues is finite.

4 Number of Points in Pfaffian Point Fields

Proof of Theorem 1.8 The moments of the number of points distribution can be written as polynomials in traces of the kernel and its powers. Therefore, they can be written as symmetric polynomials of the kernel eigenvalues. It follows that the characteristic function of $\mathcal{N}$ can also be written as a symmetric function of the kernel eigenvalues. The form of this function can be recovered from the particular case when the kernel has a real diagonal form and Theorem 1.5 is applicable. In this case $\mathcal{N}$ is the sum of independent Bernoulli random variables, $\mathcal{N}=\xi_1+\cdots+\xi_r$, where ξ_k equals 0 or 1 with probabilities $1-\lambda_k$ and λ_k, respectively. By properties of characteristic functions,

$$\varphi_{\mathcal{N}}(t) = \prod_{k=1}^{r}\varphi_{\xi_k}(t) = \prod_{k=1}^{r}\big(1+\big(e^{it}-1\big)\lambda_k\big).$$

The second equality follows because the determinant can be written as the product of eigenvalues. □

Proof of Theorem 1.9 The expression for the characteristic function of $\mathcal{N}_n$ is

$$\varphi_{\mathcal{N}_n}(t) = \prod_{k=1}^{r_n}\big(1+\big(e^{it}-1\big)\lambda_k^{(n)}\big),$$

where $\lambda_k^{(n)}$ are eigenvalues of the kernel K_n. This expression is the same as for the sum of independent Bernoulli random variables $\xi_k^{(n)}$ with $\mathbb{P}(\xi_k^{(n)}=1)=\lambda_k^{(n)}$. By the Lindenberg-

Feller theorem ([19], Theorem III.4.2 on p. 334) we conclude that if $\mathbb{V}ar(\mathcal{N}_n) \to \infty$ as $n \to \infty$, then

$$\frac{\mathcal{N}_n - \mathbb{E}(\mathcal{N}_n)}{\sqrt{\mathbb{V}ar(\mathcal{N}_n)}}$$

approaches the standard Gaussian random variable. □

5 Example: Symplectic Ensembles of Random Matrices

5.1 Circular Symplectic Ensemble

The circular symplectic ensemble of random matrices (CUE) is defined as the probability space of N-by-N self-dual real quaternion unitary matrices. The probability space has the Haar measure.

The eigenvalues of matrices from this ensembles are located on the unit circle and can be identified with angles θ_k. The density of the eigenvalue distribution is

$$c \prod_{1 \le j < k \le N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^4, \quad -\pi \le \theta_l < \pi.$$

(For more details see Dyson's papers or Chap. 2 in Forrester's book [8].)

Let N be any positive integer. Define

$$s_{2N}(\theta) := \frac{1}{2\pi} \sum_p e^{ip\theta} = \frac{1}{\pi} \sum_{p>0} \cos(p\theta) = \frac{1}{2\pi} \frac{\sin(N\theta/2)}{\sin(\theta/2)}.$$

(Here the first summation is over $p = (-2N+1)/2, (-2N+3)/2, \ldots, (2N-1)/2$, and the second summation is over $p = 1/2, \ldots, (2N-1)/2$.) Note that $s_{2N}(\theta)$ is even in θ.

Following Dyson, we write

$$Ds_{2N}(\theta) := \frac{d}{d\theta} s_{2N}(\theta) = \frac{i}{2\pi} \sum_p p e^{ip\theta} = -\frac{1}{\pi} \sum_{p>0} p \sin(p\theta),$$

and

$$Is_{2N}(\theta) := \int_0^\theta s_{2N}(\theta') d\theta',$$

so that

$$Is_{2N}(\theta) = \frac{1}{2\pi i} \sum_p p^{-1} e^{ip\theta} = \frac{1}{\pi} \sum_{p>0} \frac{1}{p} \sin(p\theta)$$

The functions Ds_{2N}, and Is_{2N} are odd in θ.

Define the quaternion function $\sigma_{N4}(\theta)$ by its matrix representation:

$$\varphi(\sigma_{N4}(\theta)) = \frac{1}{2} \begin{pmatrix} s_{2N}(\theta) & Ds_{2N}(\theta) \\ Is_{2N}(\theta) & s_{2N}(\theta) \end{pmatrix} = \frac{1}{2\pi} \sum_{p>0} \begin{pmatrix} \cos(p\theta) & -p\sin(p\theta) \\ p^{-1}\sin(p\theta) & \cos(p\theta) \end{pmatrix}.$$

In terms of quaternions, the kernel can be written as follows:

$$\sigma_{N4}(\theta) = \frac{1}{2}\left(s_{2N} - \frac{1}{2}(Is_{2N} + Ds_{2N})i\mathbf{i} + \frac{1}{2}(Is_{2N} - Ds_{2N})\mathbf{j}\right)$$
$$= \frac{1}{2\pi}\sum_{p=1/2}^{N-1/2}(\cos p\theta + a_p \sin p\theta),$$

where

$$a_p = \frac{1}{2p}\left[\left(p^2-1\right)i\mathbf{i} + \left(p^2+1\right)\mathbf{j}\right].$$

It is easy to check that $a_p^2 = -1$.

Dyson proved the following result (See Theorem 3 in [7]): The random field of eigenvalues is Pfaffian with the kernel $\sigma_{N4}(\theta - \theta')$.

The kernel σ_{N4} is a projection on a subspace of a finite-dimensional linear space (or rather module) L over $\mathbb{Q}$.

$$L = \operatorname{span}\left\{\frac{1}{\sqrt{\pi}}\cos p\theta, \frac{1}{\sqrt{\pi}}\sin p\theta, p = \frac{1}{2}, \frac{3}{2}, \dots, \frac{2N-1}{2}\right\}.$$

Note that $\dim_{\mathbb{Q}} L = 2N$.

If we calculate the action of the kernel $\sigma_{N4}(\theta)$ in the basis $\pi^{-1/2}\cos p\theta$, $\pi^{-1/2}\sin p\theta$, then we find the following matrix representation of the operator with kernel σ_{N4}:

$$K = \frac{1}{2}\left\{\begin{pmatrix} 1 & a_{1/2} \\ -a_{1/2} & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & a_{3/2} \\ -a_{3/2} & 1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & a_{N-1/2} \\ -a_{N-1/2} & 1 \end{pmatrix}\right\}. \tag{20}$$

That is, K is a $2N$-by-$2N$ block-diagonal matrix that has blocks

$$\frac{1}{2}\begin{pmatrix} 1 & a_p \\ -a_p & 1 \end{pmatrix}$$

on its main diagonal.

It follows that it can be written as

$$K = \sum_p v_p v_p^*,$$

where

$$v_p^* = \frac{1}{\sqrt{2}}(0, \dots, 0, 1, a_p, 0, \dots, 0),$$

and the entries 1 and a_p are at places $2p$ and $2p+1$, respectively.

Hence the kernel $\sigma_{N4}(\theta)$ has a quasi-real diagonal form with eigenvalues $\lambda_p = 1$.

Clearly the restriction of this random point field to an interval $I = (a, b)$ is also a Pfaffian random point field with the kernel $\sigma_{IN4} = \mathbf{1}_I(\theta)\sigma_{N4}(\theta - \theta')\mathbf{1}_I(\theta')$. Numerical evaluations suggest that this restricted kernel is also quasi-real with positive eigenvalues for arbitrary N and I. However, the proof of this claim is elusive. The author proved it only for $N = 2$.

5.2 Gaussian Symplectic Ensemble

The Gaussian symplectic ensembles of random matrices (GSE) consists of N-by-N self-dual real quaternion matrices H with the density given by the formula

$$p(H) = c \exp\left[-2\mathrm{Tr}\left(H^2\right)\right].$$

The ensemble is called Gaussian because the entries of H have components that are real Gaussian variables with variance $1/4$.

The eigenvalues of a GSE matrix are real and have the density

$$c' \prod_{j<k} (x_j - x_k)^4 \prod_{j=1}^{N} w(x_j) dx_j,$$

where $w(x) = \exp(-x^2)$. Let $Q_j(x)$ be polynomials of degree j which are orthogonal with respect to weight $w(x)$. That is, :

$$\langle Q_{2j}, Q_{2j+1} \rangle := \int \left(Q_{2j}(x) Q'_{2j+1}(x) - Q'_{2j}(x) Q_{2j+1}(x)\right) w(x) dx = 1$$

$\langle Q_{2j+1}, Q_{2j} \rangle = -1$ and $\langle Q_k, Q_l \rangle = 0$ for all other choices of k and l.

Define

$$S_N(x, y) = \sqrt{w(x)w(y)} \sum_{k=0}^{2N-1} \left[Q'_{2k+1}(x) Q_{2k}(y) - Q'_{2k}(x) Q_{2k+1}(y)\right],$$

$$I_N(x, y) = \sqrt{w(x)w(y)} \sum_{k=0}^{2N-1} \left[Q_{2k+1}(x) Q_{2k}(y) - Q_{2k}(x) Q_{2k+1}(y)\right],$$

$$D_N(x, y) = \sqrt{w(x)w(y)} \sum_{k=0}^{2N-1} \left[-Q'_{2k+1}(x) Q'_{2k}(y) + Q'_{2k}(x) Q'_{2k+1}(y)\right],$$

Define the quaternion kernel $K_N(x, y)$ by its matrix representation:

$$\varphi\left(K_N(x, y)\right) = \begin{pmatrix} S_N(x, y) & D_N(x, y) \\ I_N(x, y) & S_N(y, x) \end{pmatrix}.$$

The eigenvalues of GSE form a Pfaffian field with this kernel (see, for example, Tracy and Widom [24] or Chap. 5 in Mehta [16] for an explanation).

We can introduce the quaternion functions $\chi_k(x)$ as follows:

$$\varphi\left(\chi_k(x)\right) = \sqrt{w(x)} \begin{pmatrix} Q_{2k}(x) & -Q'_{2k}(x) \\ -Q_{2k+1}(x) & Q'_{2k+1}(x) \end{pmatrix}.$$

The dual is

$$\varphi\left(\chi_k^*(x)\right) = \sqrt{w(x)} \begin{pmatrix} Q'_{2k+1}(x) & Q'_{2k}(x) \\ Q_{2k+1}(x) & Q_{2k}(x) \end{pmatrix},$$

and we find that

$$K_N(x, y) = \sum_{k=0}^{N-1} \chi_k^*(x) \chi_k(y).$$

Since χ_k are orthonormal, hence we find that the kernel $K_N(x,y)$ is finite rank and has a quasireal diagonal form with all eigenvalues equal to 1.

The kernel is positive since the determinants $\mathrm{Det}_M(K_N(x_i,x_j))|_{1\le i,j\le m}$ can be interpreted as correlation functions for eigenvalues. Unfortunately, it is not clear how to show the positivity without appeal to correlation functions. For this reason, Proposition 3.1 cannot be used to give an independent proof that this kernel corresponds to a valid random point field.

It is also an open question whether the restrictions of this kernel to finite intervals have positive eigenvalues. This would be necessary to show in order to establish the CLT by using Theorem 1.9. (Note however that for the Gaussian symplectic ensemble, the CLT is already known from the results in [4], which are based on the relations between eigenvalues of Gaussian symplectic, orthogonal and unitary ensembles.)

Appendix A: Quaternion Matrices and Determinants

The algebra of complex quaternions $\mathbb{Q}_{\mathbb{C}}$ is isomorphic to the algebra of two-by-two complex matrices $M_2(\mathbb{C})$, with the correspondence defined by the rules

$$\mathbf{i}=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\qquad \mathbf{j}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \text{and}\quad \mathbf{k}=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}.$$

In terms of 2-by-2 matrices, if $q=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, then its conjugate is $q^*=\left(\begin{smallmatrix}d & -b\\ -c & a\end{smallmatrix}\right)$.

A number λ is called an eigenvalue of a quaternion matrix X if for some non-zero quaternion vector v, we have $Xv=v\lambda$. (These are the right eigenvalues of X, which are the most convenient in applications.) It is easy to see that if λ is an eigenvalue, then $q^{-1}\lambda q$ is also an eigenvalue for any quaternion q. However, for self-dual real quaternion matrices, all eigenvalues are real and it is possible to show that every n-by-n matrix X of this type has exactly n eigenvalues (counting with multiplicities); see Zhang [25].

It is possible and useful to generalize the concept of determinant to quaternion matrices. There are several sensible ways to do this and in this paper we will only use the Moore-Dyson and Study determinants. Interested reader can find details in a review paper by Aslaksen [1].

If we replace each entry of a quaternion matrix X by a corresponding 2-by-2 complex matrix, then the matrix X becomes represented by a $2n$-by-$2n$ complex matrix which we denote $\varphi(X)$. Then the Study determinant of X is defined as the usual determinant of $\varphi(X)$.

$$\mathrm{Det}_S(X):=\det\bigl(\varphi(X)\bigr).$$

It can also be defined in a slightly different way for real quaternion matrices. If $X=X_1+X_2\mathbf{i}+X_3\mathbf{j}+X_4\mathbf{k}$, where X_1, X_2, X_3, and X_4 are real, then we define two complex matrices $A=X_1+X_2 i$ and $B=X_3+X_4 i$. Then, $\psi(X)$ is defined as a $2n$-by-$2n$ complex matrix $\left(\begin{smallmatrix}A & B\\ -\overline{B} & \overline{A}\end{smallmatrix}\right)$, where $\overline{A}$ is the conjugate of matrix A, that is, $(\overline{A})_{kl}=\overline{(A_{kl})}$, and similarly for $\overline{B}$. The matrix $\psi(X)$ is called the *complex adjoint* of matrix X. Then, $\mathrm{Det}_S(X)=\det(\psi(X))$.

The Study determinant is multiplicative: $\mathrm{Det}_S(AB)=\mathrm{Det}_S(A)\,\mathrm{Det}_S(B)$ for square matrices A and B.

The Moore-Dyson determinant of a self-dual real quaternion matrix X can be defined as the product of the right eigenvalues of the matrix. Remarkably, this determinant can also be extended to all quaternion matrices by using a variant of the Cayley combinatorial formula for the determinant. Namely, let S_n be the group of permutations of the set $\{1,\dots,n\}$. Write every permutation σ as a product of cycles:

$$\sigma = (n_1 i_2 \cdots i_s)(n_2 j_2 \cdots j_t)\cdots(n_r k_2 \cdots k_l),$$

where n_i are the largest elements of each cycle and $n_1 > n_2 > \cdots > n_r$. Then we can write

$$\mathrm{Det}_M(X) = \sum_\sigma \varepsilon(\sigma)(X_{n_1 i_2} X_{i_2 i_3} \cdots X_{i_s n_1})\cdots(X_{n_r k_2} X_{k_2 k_3} \cdots X_{k_l n_r}),$$

where $\varepsilon(\sigma) = (-1)^{n-r}$ is the sign of the permutation σ. (see [17] and [7]).

Note that this definition allows one to calculate the quantity $\mathrm{Det}_M(X)$ for an arbitrary quaternion matrix. Dyson established that this quantity is scalar for every self-dual matrix, that is, in this case the **i**, **j**, and **k** components of the determinant are zero.

The Moore-Dyson quaternion determinant of a self-dual quaternion matrix can also be written as the Pfaffian of a related complex matrix. Let J be a $2n$-by-$2n$ block-diagonal matrix with the blocks $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ on the main diagonal. If X is a self-dual quaternion matrix, then $-J\varphi(X)$ is antisymmetric (that is, $[-J\varphi(X)]^T = J\varphi(X)$), and we can compute the Pfaffian of this matrix. We have

$$\mathrm{Det}_M(X) = \mathrm{Pf}\big(-J\varphi(X)\big). \tag{21}$$

(see [7] and Proposition 6.1.5 on p. 238 in Forrester's book [8]).

In terms of the transformation ψ, this can be written as follows. Let X be a real quaternion matrix and let A and B be defined as in the definition of the complex adjoint. Let $\widetilde{J} = \psi(\mathbf{j}) = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. Then, $-\widetilde{J}\psi(X) = \left(\begin{smallmatrix} \overline{B} & \overline{A} \\ -A & B \end{smallmatrix}\right)$. If the real quaternion matrix X is self-dual, then $-\widetilde{J}\psi(X)$ is antisymmetric, and it can be shown that

$$\mathrm{Det}_M(X) = -\mathrm{Pf}\big(-\widetilde{J}\psi(X)\big). \tag{22}$$

The Study and Moore-Dyson determinants are related by the following formula:

$$\mathrm{Det}_S(X) = \mathrm{Det}_M\big(X^* X\big). \tag{23}$$

(See formula (6.13) on page 239 in [8] or Corollary 5.1.3 on p. 75 in [16].)

Appendix B: Proof of Theorem 2.1

Since the identity is algebraic, it is enough to show that it holds for matrices with real quaternion entries. We will prove this by showing that the corresponding result holds if we write the quaternion determinants in terms of Pfaffians. Namely, let $C = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k}$ and define complex matrices $A = X_1 + X_2 i$ and $B = X_3 + X_4 i$. Then, by using (22) we obtain that

$$\mathrm{Det}_M\big(C^*C\big) = -\mathrm{Pf}\begin{pmatrix} -B^*A + (B^*A)^t & (A^*A)^t + B^*B \\ -A^*A - (B^*B)^t & A^*B - (A^*B)^t \end{pmatrix}.$$

The blocks $-B^*A + (B^*A)^t$ and $A^*B - (A^*B)^t$ are antisymmetric, and $((A^*A)^t + B^*B)^t = -A^*A - (B^*B)^t$, so the block matrix is antisymmetric as well.

What we need to prove is that

$$\operatorname{Pf}\begin{pmatrix} -B^*A+(B^*A)^t & (A^*A)^t+B^*B \\ -A^*A-(B^*B)^t & A^*B-(A^*B)^t \end{pmatrix}$$
$$=\sum_I \operatorname{Pf}\begin{pmatrix} -B^{I*}A^I+(B^{I*}A^I)^t & (A^{I*}A^I)^t+B^{I*}B^I \\ -A^{I*}A^I-(B^{I*}B^I)^t & A^{I*}B^I-(A^{I*}B^I)^t \end{pmatrix},$$

where the summation is over all ordered m-tuples $I=(i_1<\cdots<i_m)$, with $i_k\in\{1,\dots,n\}$, and A^I, B^I are the matrices that are obtained from matrices A and B, respectively, by taking the rows with indices in I.

In order to prove this, we recall that if R is a $2m$-by-$2m$ antisymmetric matrix, then the Pfaffian of R is defined as follows:

$$\operatorname{Pf}(R)=\frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\operatorname{sgn}(\sigma)\prod_{i=1}^m R_{\sigma(2i-1)\sigma(2i)}. \tag{24}$$

Next, we note that if

$$R=\begin{pmatrix} -B^*A+(B^*A)^t & (A^*A)^t+B^*B \\ -A^*A-(B^*B)^t & A^*B-(A^*B)^t \end{pmatrix},$$

then there is a formula for R_{ij} in terms of elements of A and B. This formula depends on whether i and j are greater or less than m. For example if i and j are both $\le m$, then

$$R_{ij}=\sum_{a=1}^n(-\overline{B}_{a,i}A_{a,j}+\overline{B}_{a,j}A_{a,i})\equiv\sum_{a=1}^n\Psi_a(i,j).$$

If we substitute this in formula (24), and expand, then we get

$$\operatorname{Pf}(R)=\frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\operatorname{sgn}(\sigma)\sum_{(a,b,\dots,z)}\Psi_a\big(\sigma(1),\sigma(2)\big)\cdots\Psi_z\big(\sigma(2m-1),\sigma(2m)\big), \tag{25}$$

where the summation is over all m-tuples $(a,b,\dots,z)$ with each letter taking a value in $\{1,\dots,n\}$.

A similar formula holds for the Pfaffian of R^I, where

$$R^I=\begin{pmatrix} -B^{I*}A^I+(B^{I*}A^I)^t & (A^{I*}A^I)^t+B^{I*}B^I \\ -A^{I*}A^I-(B^{I*}B^I)^t & A^{I*}B^I-(A^{I*}B^I)^t \end{pmatrix}.$$

Namely,

$$\operatorname{Pf}\big(R^I\big)=\frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\operatorname{sgn}(\sigma)\sum_{(a,b,\dots,z)\in I^m}\Psi_a\big(\sigma(1),\sigma(2)\big)\cdots\Psi_z\big(\sigma(2m-1),\sigma(2m)\big). \tag{26}$$

The difference with the previous formula is that the elements of m-tuples $(a,b,\dots,z)$ are now restricted to take values among indices in m-tuple I.

Let us for shortness write $\Psi_{a,\dots,z}(\sigma)$ for the product $\Psi_a(\sigma(1),\sigma(2))\cdots\Psi_z(\sigma(2m-1),\sigma(2m))$.

If all elements of $(a, b, \dots, z)$ are different, then the term $\Psi_{a,\dots,z}(\sigma)$ occurs once in expansion (25) and once in the sum of expansions (26),

$$\sum_I \mathrm{Pf}\bigl(R^I\bigr).$$

(In this sum, it occurs in the expansion of that $\mathrm{Pf}(R^I)$, for which I is the ordered version of the m-tuple $(a, b, \dots, z)$.)

If some of the elements of $(a, b, \dots, z)$ coincide, then the situation is different. The term $\Psi_{a,\dots,z}(\sigma)$ occurs once in expansion (25) but it can occur more than once in the sum

$$\sum_I \mathrm{Pf}\bigl(R^I\bigr).$$

For example, if all elements of the m-tuples are the same, $a = b = \cdots = z$, then this term will appear in the expansion of each $\mathrm{Pf}(R^I)$, whose index I contain a.

Clearly, in order to prove that $\mathrm{Pf}(R) = \sum_I \mathrm{Pf}(R^I)$, it is enough to prove that the sum of all these terms is zero. That is, it is enough to show that for a fixed m-tuple $(a, b, \dots, z)$ with at least two elements that are equal, the sum

$$\sum_{\sigma \in S_{2m}} \mathrm{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma)$$

is zero.

Without loss of generality we can assume that $a = b$. We have to consider several cases of σ, which are summarized in the following table

	$\sigma(1)$	$\sigma(2)$	$\sigma(3)$	$\sigma(4)$
$S_{2m}[1]$				
$S_{2m}[2]$			*	*
$S_{2m}[3]$		*		*
$S_{2m}[4]$		*	*	
$S_{2m}[5]$				*
$S_{2m}[6]$		*		
$S_{2m}[7]$			*	
$S_{2m}[8]$		*	*	*

The star means that the corresponding $\sigma(i)$ is greater than m. For example, $S_{2m}[1]$ denote the set of all permutations from S_{2m} that satisfy the condition that all of $\sigma(1), \sigma(2), \sigma(3)$, $\sigma(4)$ are smaller than or equal to m. For permutations in this set,

$$\Psi_a\bigl(\sigma(1), \sigma(2)\bigr)\Psi_a\bigl(\sigma(3), \sigma(4)\bigr) = (-\overline{B}_{a,i}A_{a,j} + \overline{B}_{a,j}A_{a,i})(-\overline{B}_{a,k}A_{a,l} + \overline{B}_{a,l}A_{a,k}).$$

$S_{2m}[2]$ denote the set of all permutations from S_{2m} that satisfy the condition that $\sigma(1), \sigma(2)$ are smaller than or equal to m and $\sigma(3), \sigma(4)$ are greater than m, and so on.

Let us define $\tau_1[\sigma]$, as a permutation that coincides with σ on all indices except 2 and 4, for which it is defined by equalities $\tau_1[\sigma](2) = \sigma(4)$, and $\tau_1[\sigma](4) = \sigma(2)$. Similarly, $\tau_2[\sigma]$ is defined as a permutation which acts on everything except 2 and 3 as σ, and on these indices it is defined by $\tau_2[\sigma](2) = \sigma(3)$, $\tau_2[\sigma](3) = \sigma(2)$. Finally we define $\tau_3[\sigma]$ as a permutation that coincides with σ on all indices except 2, 3, and 4, where it is defined by the rules: $\tau_3[\sigma](2) = \sigma(4)$, $\tau_3[\sigma](3) = \sigma(2)$, $\tau_3[\sigma](4) = \sigma(3)$. Note that $\mathrm{sgn}(\tau_1[\sigma]) = \mathrm{sgn}(\tau_2[\sigma]) = -\mathrm{sgn}(\sigma)$, and $\mathrm{sgn}(\tau_3[\sigma]) = \mathrm{sgn}(\sigma)$. Observe that for an arbitrary function f,

$$\sum_{\sigma \in S_{2m}[1]} \big(f(\sigma) + f\big(\tau_1[\sigma]\big) + f\big(\tau_2[\sigma]\big)\big) = 3 \sum_{\sigma \in S_{2m}[1]} f(\sigma).$$

It is easy to check that the identity

$$\begin{aligned} 0 = {} & \Psi_a\big(\sigma(1), \sigma(2)\big)\Psi_a\big(\sigma(3), \sigma(4)\big) - \Psi_a\big(\tau_1[\sigma](1), \tau_1[\sigma](2)\big)\Psi_a\big(\tau_1[\sigma](3), \tau_1[\sigma](4)\big) \\ & -\Psi_a\big(\tau_2[\sigma](1), \tau_2[\sigma](2)\big)\Psi_a\big(\tau_2[\sigma](3), \tau_2[\sigma](4)\big) \end{aligned}$$

holds for permutations in $S_{2m}[1]$, and this implies that

$$\sum_{\sigma \in S_{2m}[1]} \mathrm{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma) = 0. \tag{27}$$

Next,

$$\sum_{\sigma \in S_{2m}[2]} \big(f(\sigma) + f\big(\tau_1[\sigma]\big) + f\big(\tau_3[\sigma]\big)\big) = \sum_{\sigma \in S_{2m}[2]} f(\sigma) + 2 \sum_{\sigma \in S_{2m}[3]} f(\sigma).$$

and it is easy to check the identity

$$\begin{aligned} 0 = {} & \Psi_a\big(\sigma(1), \sigma(2)\big)\Psi_a\big(\sigma(3), \sigma(4)\big) - \Psi_a\big(\tau_1[\sigma](1), \tau_1[\sigma](2)\big)\Psi_a\big(\tau_1[\sigma](3), \tau_1[\sigma](4)\big) \\ & + \Psi_a\big(\tau_3[\sigma](1), \tau_3[\sigma](2)\big)\Psi_a\big(\tau_3[\sigma](3), \tau_3[\sigma](4)\big) \end{aligned} \tag{28}$$

for $\sigma \in S_{2m}[2]$. Hence, identity (28) implies that

$$\Bigg(\sum_{\sigma \in S_{2m}[2]} + 2 \sum_{\sigma \in S_{2m}[3]}\Bigg) \mathrm{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma) = 0 \tag{29}$$

The other cases are similar. We use

$$\sum_{\sigma \in S_{2m}[4]} \big(f(\sigma) + f\big(\tau_1[\sigma]\big) + f\big(\tau_3[\sigma]\big)\big) = 2 \sum_{\sigma \in S_{2m}[4]} f(\sigma) + \sum_{\sigma \in S_{2m}[2]} f(\sigma)$$

in order to conclude that

$$\Bigg(2 \sum_{\sigma \in S_{2m}[4]} + \sum_{\sigma \in S_{2m}[2]}\Bigg) \mathrm{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma) = 0 \tag{30}$$

By adding (29) and (30), we obtain

$$\Bigg(\sum_{\sigma \in S_{2m}[2]} + \sum_{\sigma \in S_{2m}[3]} + \sum_{\sigma \in S_{2m}[4]}\Bigg) \mathrm{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma) = 0. \tag{31}$$

Next, the identity

$$\sum_{\sigma \in S_{2m}[5]} \big(f(\sigma) + f\big(\tau_1[\sigma]\big) + f\big(\tau_3[\sigma]\big)\big) = 2 \sum_{\sigma \in S_{2m}[5]} f(\sigma) + \sum_{\sigma \in S_{2m}[6]} f(\sigma)$$

implies that

$$\left(2\sum_{\sigma\in S_{2m}[5]}+\sum_{\sigma\in S_{2m}[6]}\right)\operatorname{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma)=0, \tag{32}$$

and the identity

$$\sum_{\sigma\in S_{2m}[6]}\left(f(\sigma)+f\big(\tau_1[\sigma]\big)+f\big(\tau_3[\sigma]\big)\right)=\sum_{\sigma\in S_{2m}[6]}f(\sigma)+2\sum_{\sigma\in S_{2m}[7]}f(\sigma)$$

implies that

$$\left(\sum_{\sigma\in S_{2m}[6]}+2\sum_{\sigma\in S_{2m}[7]}\right)\operatorname{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma)=0. \tag{33}$$

By adding (32) and (33), we obtain:

$$\left(\sum_{\sigma\in S_{2m}[5]}+\sum_{\sigma\in S_{2m}[6]}+\sum_{\sigma\in S_{2m}[7]}\right)\operatorname{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma)=0. \tag{34}$$

Finally,

$$\sum_{\sigma\in S_{2m}[8]}\left(f(\sigma)+f\big(\tau_1[\sigma]\big)+f\big(\tau_3[\sigma]\big)\right)=3\sum_{\sigma\in S_{2m}[8]}f(\sigma).$$

Identity (28) still holds in this case, and therefore,

$$\sum_{\sigma\in S_{2m}[8]}\operatorname{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma)=0. \tag{35}$$

Clearly, the sets $S_{2m}[k]$, are disjoint and their union over $k=1,\dots,8$ consists of all permutations from S_{2m} for which $\sigma(1)\le m$. Hence, identities (27), (31), (34), and (35) imply that

$$\sum_{\substack{\sigma\in S_{2m}\\ \sigma(1)\le m}}\operatorname{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma)=0.$$

The case $\sigma(1)>m$ can be handled similarly, and we obtain

$$\sum_{\sigma\in S_{2m}}\operatorname{sgn}(\sigma)\Psi_{a,\dots,z}(\sigma)=0.$$

This holds provided the first two indices in $(a,\dots,z)$ are equal. It is clear that this identity also holds if any two indices in $(a,\dots,z)$ are equal.

As observed earlier, this implies that we can remove all terms $\Psi_{a,\dots,z}(\sigma)$, for which two indices in $(a,\dots,z)$ coincide, from the expansions of both $\operatorname{Pf}(R)$ and the sum $\sum_I \operatorname{Pf}(R^I)$. (See expansions (25) and (26).) Then it is clear that the remaining terms in the expansions are the same. Hence

$$\operatorname{Pf}(R)=\sum_I \operatorname{Pf}\big(R^I\big),$$

and this implies the statement of the theorem.

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