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A 3D Ginibre Point Field

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Abstract We introduce a family of three-dimensional random point fields using the concept of the quaternion determinant. The kernel of each field is an n-dimensional orthogonal projection on a linear space of quaternionic polynomials. We find explicit formulas for the basis of the orthogonal quaternion polynomials and for the kernel of the projection. For number of particles $n \to \infty$, we calculate the scaling limits of the point field in the bulk and at the center of coordinates. We compare our construction with the previously introduced Fermi-sphere point field process.

Keywords Random point fields · Ginibre ensemble · Determinantal point field · Fermi sphere field · Quaternions · Fermionic point process

1 Introduction

A random point field in a space X is a configuration of points in X sampled according to a joint probability distribution on the space of all configurations. An important class of random point fields are determinantal fields, for which this joint distribution can be written in terms of determinants of a kernel function [4,12,13].

While the general theory of determinantal fields is well-developed [14], the study of concrete realizations focused mostly on point fields in dimensions 1 and 2. These fields often arise as eigenvalues of random matrices, or zeros of random analytic functions [8].

The studies of determinantal processes in higher dimensions are less numerous. One notable example is a recent paper [16] which introduced a determinantal point field called *the Fermi sphere* ensemble. This field is defined for an arbitrary dimension of the ambient space.

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The Fermi sphere field has an infinite number of points and its kernel is modeled on the one-dimensional sine-kernel,

$$K(s,t)=\frac{\sin\pi(s-t)}{\pi(s-t)}, \tag{1}$$

which can also be written as $\hat{\mathbb{1}}_1(s-t)$, where $\hat{\mathbb{1}}_1$ is the Fourier transform of the indicator function $\mathbb{1}_{[-1/2,1/2]}$.

Likewise, the kernel of the d-dimensional Fermi sphere process is defined as $\hat{\mathbb{1}}_d(\vec{s}-\vec{t})$, where $\hat{\mathbb{1}}_d$ is the Fourier transform of the indicator function of a d-dimensional ball. This kernel is positive definite, and the other required properties are achieved by a suitable choice of the radius of the ball, R.

By this method, $K(\vec{s},\vec{t})=\hat{\mathbb{1}}_d(|\vec{s}-\vec{t}|)$, with

$$\hat{\mathbb{1}}_d(x)=R^d\frac{J_{d/2}(2\pi Rx)}{(Rx)^{d/2}}.$$

Here $J_{d/2}(x)$ is the Bessel function and

$$R=\left[\frac{\Gamma(1+d/2)}{\sqrt{\pi}}\right]^{1/d}.$$

For $d=1$, this expression recovers the kernel in (1). For $d=2$, it gives

$$\hat{\mathbb{1}}_2(x)=\frac{J_1(2\pi^{3/4}x)}{\pi^{3/4}x}, \tag{2}$$

and for $d=3$,

$$\hat{\mathbb{1}}_3(x)=-3\,\frac{t\cos t-\sin t}{t^3},\ \text{ with } t=2\pi\left(\frac{3}{4}\right)^{1/3}x. \tag{3}$$

For $d=2$, we can compare the Fermi sphere field with another well-known ensemble, the Ginibre random point field. A finite Ginibre field with n points is formed by eigenvalues of an $n\times n$ non-Hermitian complex Gaussian random matrix [7] and its kernel is

$$\mathcal{K}_n(z,w)=\frac{1}{\pi}e^{-\frac{1}{2}\left(|z|^2+|w|^2\right)}\sum_{k=0}^{n-1}\frac{(z\overline{w})^k}{k!}. \tag{4}$$

If $n\to\infty$, then we obtain the infinite Ginibre field with the kernel

$$\mathcal{K}_\infty(z,w)=\frac{1}{\pi}e^{-\frac{1}{2}\left(|z|^2+|w|^2\right)+z\overline{w}} \tag{5}$$

$$=\frac{1}{\pi}e^{-\frac{1}{2}|z-w|^2+i\,\mathrm{Im}\,z\overline{w}}. \tag{6}$$

The infinite Ginibre point field is quite remarkable. The correlation functions are translation invariant. Indeed, if we make a substitution $z'=z+t$, and $w'=w+t$, then we find that $\mathcal{K}_\infty(z',w')=\mathcal{K}_\infty(z,w)\exp(i\,\mathrm{Im}\,(t\overline{w}+z\overline{t}))$, and a calculation shows that

$$\det\left|\mathcal{K}(z_i',z_j')\right|_{i,j=1}^n=\det\left|\mathcal{K}(z_i,z_j)\right|_{i,j=1}^n\times\exp\left(i\,\mathrm{Im}\left(t\sum_{i=1}^n\overline{z_i}+\overline{t}\sum_{i=1}^n z_i\right)\right)$$
$$=\det\left|\mathcal{K}(z_i,z_j)\right|_{i,j=1}^n.$$

Moreover, this field is among a small number of fields which are invariant relative to a large group of complex plane transformations (See Krishnapur [11].).

The infinite Ginibre field is clearly different from the 2-dimensional Fermi sphere field and its properties are apparently nicer. Can we extend its definition to dimensions greater than 2?

One way to describe a determinantal random point field on a measure space X is to relate it to a closed linear subspace V of $L^2(X, \mu)$, the Hilbert space of square integrable functions on X (See, for example, [8].). Indeed, we can define correlations of the corresponding field as

$$R_k(x_1, \dots, x_k) = \mathrm{Det}[K(x_i, x_j)]\big|_{1 \le i, j \le k}, \text{ for all } k \in \mathbb{N},$$

where $K(x, y)$ is the kernel of the orthogonal projection $K\colon L^2(X, \mu) \to V$,

$$(Kf)(x) = \int_X K(x, y) f(y) \mu(dy).$$

For example, if $V = L^2(X, \mu)$, then the resulting field is the Poisson field with the intensity μ. The Fermi sphere field defined above corresponds to the subspace of functions with Fourier coefficients that vanish outside of a d-dimensional ball.

From this prospective, the classical Ginibre field corresponds to the space V_n of all complex valued polynomials of degree no larger than $n-1$. The measure μ here is the standard Gaussian measure on the complex plane.

Our goal in this paper is to design an analogous field on $X = \mathbb{R}^3$, which we identify with the space of pure imaginary quaternions. The field is not determinantal but rather quaternion-determinantal since it uses the quaternion determinant in the definition of its correlation functions. However, we are going to use a construction similar to the construction of the Ginibre field. It starts with a linear subspace V of quaternion polynomials on X, defines a projection kernel, and results in a quaternion determinantal field associated to V.

The kernel of this field $K_n(z, w)$ can be written as

$$K_n(z, w) = \sum_{k=0}^{n-1} \frac{P_k(z)\overline{P_k(w)}}{h_k}, \tag{7}$$

where z and w are purely imaginary quaternions, identified with points in $\mathbb{R}^3$. Polynomials $P_k(z) \in \mathbb{H}[z]$ are the monic quaternion polynomials of degree k, which are orthogonal with respect to the standard Gaussian measure μ on $\mathbb{R}^3$,

$$f(z) = (2\pi)^{-3/2} e^{-|z|^2/2}. \tag{8}$$

(We will also use the functions $\mathcal{K}_n(z, w) = K_n(z, w)\sqrt{f(z) f(w)}$.)

We have to be careful both about the concept of quaternion polynomials and the notion of their orthogonality, since the quaternions are not commutative.[1] For this reason, we consider the most parsimonious definition of quaternion polynomials as the right $\mathbb{H}$-module $\mathcal{M}$ generated by the monomials z^k, $k = 0, 1, \dots$. In addition we restrict attention to the case when the arguments of the polynomials belong to the space of imaginary quaternions Λ. Elements of the module $\mathcal{M}$ are finite sums of $z^k a_k$, where a_k are arbitrary quaternions and $z \in \Lambda$,

$$P(z) = \sum_{k=0}^{n} z^k a_k.$$

This module is still a vector space over $\mathbb{R}$.

[1] Some background material about quaternions is collected in Appendix A.

Let

$$d\mu(z) = f(z)\, dm(z),$$

where $dm(z)$ is the Lebesgue measure on $\mathbb{R}^3$, and define the following $\mathbb{R}$-bilinear map $\mathcal{M} \times \mathcal{M} \to \mathbb{H}$,

$$\langle u(z), w(z) \rangle = \int_\Lambda \overline{u}(z) w(z)\, \mu(dz). \tag{9}$$

This is a scalar product if we consider $\mathcal{M}$ as $\mathbb{R}$-vector spaces. In particular, this bi-product is positive definite:

$$\langle u(z), u(z) \rangle \geq 0 \text{ for all } u(z) \in \mathcal{M},$$

and equality holds only if $u(z) = 0$.

In addition, this bi-product function is $\mathbb{H}$-linear in the second argument :

$$\langle u(z), w(z)\alpha \rangle = \langle u(z), w(z) \rangle\, \alpha, \text{ for every } \alpha \in \mathbb{H},$$

and it also has the property that $\langle w(z), u(z) \rangle = \overline{\langle u(z), w(z) \rangle}$.

As usual, the norm of a function is defined as the square root of the scalar product of the function with itself,

$$\|u(z)\| = \sqrt{\langle u(z), u(z) \rangle}.$$

We say that a family of quaternionic functions on Λ, $\{u_k\}$ is *orthonormal*, if

$$\langle u_k, u_l \rangle = \begin{cases} 1, & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases} \tag{10}$$

We define $P_k(z)$ to be the system of monic orthogonal polynomials. This system exists and unique by the analogue of the Gram-Schmidt orthogonalization process, and every element in $\mathcal{M}$ can be uniquely written as $\sum_{k=0}^{n} P_k(z) a_k$.

Finally, the constants h_k in Formula (7) are chosen to ensure that every polynomials $P_k(z)/\sqrt{h_k}$ has the unit norm, that is, $h_k = \|P_k(z)\|^2$.

Our main goal is to show that kernel in (7) can be used to define a random point field on $\mathbb{R}^3$, and study the properties of this field.

The rest of the paper is organized as follows. In Sect. 2 we give an overview of main results. In Sect. 3 we briefly review the theory of quaternion determinantal fields. In Sect. 4 we study the orthogonal polynomials P_k and the norm constants h_k. Section 5 is devoted to the study of properties of the kernel. And Sect. 6 concludes with some open problems.

2 Brief Description of the Results

First, an analogue of the Christoffel–Darboux formula holds for the kernel $K_n(z, w)$, as formulated in Theorem 5.3.[2]

Theorem 5.3 (Explicit formula for the kernel) *Let $x = us$ and $y = vt$, where $s = |x|$ and $t = |y|$. Suppose $s \neq t$. Then*

$$K_n(us, vt) = \rho_n(s, t)\frac{1 - uv}{2} + (-1)^n \delta_n(s, t)\frac{1 + uv}{2},$$

[2] The numbering of theorems refers to the section where they are proved. For example, this theorem is restated and proved in Sect. 6.

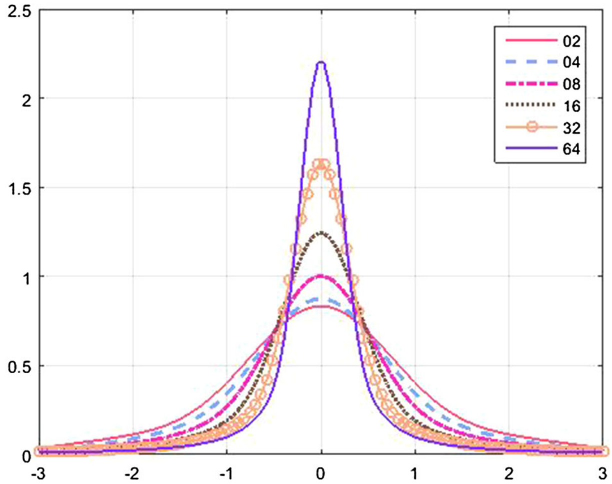

Fig. 1 Plot of $\frac{1}{n+1}K_n(is, is)e^{-s^2/2}$ for $n = 2, 4, \ldots, 64$

where

$$\rho_n(s,t) = \frac{Q_n(s)Q_{n+1}(t) - Q_{n+1}(s)Q_n(t)}{h_n(t-s)}, \text{ and}$$
$$\delta_n(s,t) = \frac{Q_n(s)Q_{n+1}(t) + Q_{n+1}(s)Q_n(t)}{h_n(t+s)}$$

and $Q_n(s) = \mathbf{i}^{-n}P_n(\mathbf{i}s)$.

From this expression for the kernel, we can derive formulae for the first and second correlation functions.

Let

$$\rho_n(s) := \lim_{t\to s}\rho_n(s,t) = \frac{1}{h_n}\big[Q_n(s)Q'_{n+1}(s) - Q_{n+1}(s)Q'_n(s)\big],$$
$$\delta_n(s) := \lim_{t\to s}\delta_n(s,t) = \frac{1}{h_n}\frac{Q_n(s)Q_{n+1}(s)}{s}.$$

Corollary 2.1 *Let $x = us$ and $y = vt$, where $s = |x|$ and $t = |y|$. Then, the first correlation function (i.e., the density of the point field with respect to the background measure) is*

$$p_n^{(1)}(x) = \rho_n(s).$$

The first correlation function is shown in Fig. 1.

Corollary 2.2 *Let $x = us$ and $y = vt$, where $s = |x|$ and $t = |y|$. The second correlation function is*

$$p_n^{(2)}(x,y) = \rho_n(s)\rho_n(t) - \left[\rho_n(s,t)^2\frac{1+\cos\alpha}{2} + \delta_n(s,t)^2\frac{1-\cos\alpha}{2}\right],$$

where α is the angle between vectors x and y.

Correlations of the points on a sphere

An interesting particular case occurs when the radii of points are equal, $t = s$. We devote next few paragraphs to correlations in this case since they are quite different here from the correlations of the classical Ginibre ensemble.

If $t = s$, then the kernel is

$$K_n(us, vs) = \frac{1}{2}\big[\rho_n(s) - \delta_n(s)\big] - \frac{1}{2}\big[\rho_n(s) + \delta_n(s)\big]uv, \tag{11}$$

and the second correlation function is

$$p_n^{(2)}(us, vs) = \big[\rho_n(s)^2 - \delta_n(s)^2\big]\frac{1 - \cos\alpha}{2}, \tag{12}$$

where α is the angle between u and v.

The kernel in (11) is multilinear in u and v with the rank 2. Hence, all determinants $\det\left([K_n(u_i s, u_j s)]_{i,j=1}^r\right)$ are zero for $r \geq 3$. It follows that all correlation functions of order higher than two vanish.

Recall that if $x_1, \ldots, x_k$ are k non-equal points, and $V_1, \ldots, V_k$ are small regions around these points, then the probability that the random field has exactly one point in each of these regions approximately equals $p_k(x_1, \ldots, x_k)\mathrm{vol}(V_1)\ldots\mathrm{vol}(V_k)$.

The vanishing of the correlation functions $p_k(x_1, \ldots, x_k)$ means that the probability to find a point in each of the regions $V_1, \ldots, V_k$ is significantly smaller than we could expect if the points were independently distributed.

In our case, this means that for the configurations conditioned to have 2 points on a given sphere, other points will behave as if there were a force that repulse them from this sphere.

This property is special for the quaternion Ginibre ensemble, and has no analog for the classical Ginibre ensemble.

It is perhaps worth mentioning that this property resembles the situation in physics, when each electronic shell in an atom can contain only a fixed number of electrons: the first shell can hold up to two electrons, the second shell can hold up to eight (2 + 6) electrons, the third shell can hold up to 18 (2 + 6 + 10) and so on.

Now let us turn to the asymptotic properties of the correlation functions when $n \to \infty$. For this purpose, let us define

$$\rho(s) := \frac{1}{(2\pi)^2}\frac{\sqrt{1 - s^2}}{s^2} \text{ for } s \in (0, 1). \tag{13}$$

Theorem 5.7 (Limit density) *Let $\rho_n(x)$ denote the density of the quaternion Ginibre field at point $x \in \mathbb{R}^3$ with respect to the Lebesque measure on $\mathbb{R}^3$. Assume that u is a pure unit quaternion and $s \in \mathbb{R}$, $s > 0$. Then,*

$$\lim_{n\to\infty} 2\sqrt{n}\rho_n\big(2\sqrt{n}us\big) = \begin{cases} \rho(s), & \text{if } s < 1, \\ 0 & \text{if } s > 1. \end{cases}$$

The theorem says that, first, with high probability all the points are contained in the ball of the radius comparable with $2\sqrt{n}$. This is similar to the classical Ginibre field. The second claim of the theorem is that the probability approaches $\frac{1}{2}n^{-1/2}\rho(s)m(V)$ for a small ball at distance $2\sqrt{ns}$ from the origin.

The total volume of the sphere of radius $2\sqrt{n}$ is $\sim n^{3/2}$. Hence the probability to find one of n field points in a particular small fixed ball V goes to 0 as $n^{-1/2}$.

Hence for large n the points in the bulk are thinly spread, and this suggest that the most interesting area is at the center of coordinates, where the function $\rho(s)$ has a singularity. We will see later that the ball around the origin with the radius $1/\sqrt{n}$ contains a finite number of points, and this is the scale at which we find interesting "at center" asymptotics.

In order to see this from a slightly different prospective, note that the quaternion point field is rotationally invariant, meaning that the correlations are invariant relative to rotations around the origin. Let the *radial density* be

$$\mathfrak{p}_n(r) = \int_{rS^2} \rho_n(x)\, d\nu(x),$$

where rS^2 is the sphere of radius r and $d\nu(x)$ is the measure on rS^2 induced by the Lebesgue measure on $\mathbb{R}^3$.

Corollary 5.8 (Limit radial density) *For the limit of radial density, we have*

$$\lim_{n\to\infty} \frac{1}{2\sqrt{n}} \mathfrak{p}_n\big(2\sqrt{n}s\big) = \begin{cases} \mathfrak{p}(s) := \frac{1}{\pi}\sqrt{1-s^2}, & \textit{if } 0 < s < 1, \\ 0 & \textit{if } s > 1, \end{cases}$$

In other words, the distribution of points across spherical shells is the classical quarter-circle distribution.

Next, we are interested in asymptotics of higher correlation functions when $n \to \infty$. First, it is easy to check that if $x \neq y$, then

$$\lim_{n\to\infty} \mathcal{K}_n(\sqrt{n}x, \sqrt{n}y) = 0.$$

That is, there are no 2-correlations at the global scale.

The asymptotics of local correlations depend on whether the points are in the balk, $|x| \approx |y| \sim \sqrt{n}$, or at the center: $|x| \approx |y| \ll \sqrt{n}$.

In order to understand the asymptotic of correlations in the bulk, we recall that the total number of field points is n and the volume of the ball of radius $\sqrt{n}$ is $\sim n^{3/2}$. Hence, we can expect that the distance between neighboring points is $\sim n^{1/6}$.

Normally, this would suggest using $n^{1/6}$ as scaling parameter. However, in our case the strength of interaction (correlation) between two points depends not on the distance between these points but rather on the difference in their absolute values and on the angle between their position vectors. For this reason, it turns out that we should scale the model by considering a spherical shell of the radius $\sim 2\sqrt{n}x_0$ and width $n^{-1/2}$.

Let

$$x_0 \in [\varepsilon, 1-\varepsilon], \text{ and } \varphi = \arccos x_0 \in (0, \pi/2). \tag{14}$$

Define

$$\begin{aligned} s_n &= 2\sqrt{n+3/2}\cos\left(\varphi + \frac{\sigma}{2\sin^2\varphi}\frac{1}{n}\right), \\ t_n &= 2\sqrt{n+3/2}\cos\left(\varphi + \frac{\tau}{2\sin^2\varphi}\frac{1}{n}\right), \end{aligned} \tag{15}$$

where σ and τ are real parameters. Here s_n and t_n represent the radii of two points. Since $\varphi \neq 0$, we have $|s_n - t_n|$ is of the order $1/\sqrt{n}$.[3]

Next, we define

$$\mathcal{K}_n\big((u,\sigma),(v,\tau);x_0\big) = \frac{2\sqrt{n+\frac{3}{2}}}{\rho(x_0)}\mathcal{K}_n(us_n, vt_n), \tag{16}$$

[3] We use factor $\sqrt{n+3/2}$ in the formula for the convenience of calculations in the proof. The factor $\sqrt{n}$ works equally well.

where u and v are pure unit quaternions, and $\rho(x_0)$ is as defined in (13).

Theorem 5.9 (Scaling limit of the kernel in the bulk) *Let $\sigma, \tau \in \mathbb{R}$ and $u, v \in S^2$, the space of pure unit quaternions. Then,*

$$\lim_{n\to\infty} \mathcal{K}_n\big((u,\sigma),(v,\tau);x_0\big) = \mathcal{K}\big((u,\sigma),(v,\tau)\big) := \frac{\sin(\tau-\sigma)}{\tau-\sigma}\frac{1-uv}{2}.$$

This gives us a kernel on the space $S^2 \times \mathbb{R}$ as a scaling limit of the quaternion kernels on $\mathbb{R}^3$.

At the origin of coordinates O, the situation is different. Let us look at the interior of the sphere $S_{r,n}$ with the center at O and radius $r/\sqrt{n}$. Since the law for the limit radial density suggests that every spherical shell of width dr contains around $\sqrt{n}\,dr$ points, we can expect that the sphere $S_{r,n}$ contains $O(r)$ points.

Let z and z' be pure quaternions. We define the limit kernel at the center as follows. Suppose $s \neq t$. Then

$$K^{(c)}(z,z') := \lim_{n\to\infty}\left(n+\frac{3}{2}\right)^{-3/2} \mathcal{K}_n\left(\frac{z}{\sqrt{n+3/2}},\frac{z'}{\sqrt{n+3/2}}\right)$$

Theorem 5.10 (Scaling limit of the kernel at the origin) *Let z and z' be pure quaternions, $z = us$ and $z' = vt$, where $s = |z|$ and $t = |z'|$. Then the limit $K^{(c)}(z,z')$ exists and*

$$\begin{aligned} K^{(c)}(us,vt) = \frac{1}{\sqrt{2\pi}}\frac{1}{st}\bigg\{&\left[\frac{\sin(t-s)}{t-s} - \frac{\sin(t+s)}{t+s}\right] \\ &-uv\left[\frac{\sin(t-s)}{t-s} + \frac{\sin(t+s)}{t+s} - 2\frac{\sin t}{t}\times\frac{\sin s}{s}\right]\bigg\}, \end{aligned}$$

Therefore, the limit density at the center is given by the function

$$\rho^{(c)}(us) \equiv \lim_{t\to s} K^{(c)}(us,ut) = \sqrt{\frac{2}{\pi}}\frac{1}{s^2}\left[1-\left(\frac{\sin s}{s}\right)^2\right].$$

The limit radial density is

$$\mathfrak{p}^{(c)}(us) = 4\sqrt{2\pi}\left[1-\left(\frac{\sin s}{s}\right)^2\right].$$

These two formulas correspond to the following picture. In a neighborhood of the origin O the random points have positive density

$$\rho^{(c)}(0) \equiv \lim_{s\to 0}\rho^{(c)}(s) = \frac{1}{3}\sqrt{\frac{2}{\pi}}.$$

Far away from the origin the density of points declines in such a way that approximately the same number of points lies on each spherical shell of equal width.

In particular, we can see that this limit random point field has an infinite number of points.

The second correlation functions is somewhat cumbersome, so we write it out in detail later and give only a graphical description here.

Figure 2 shows the correlations when one of the points is fixed at the distance $s = 1$ from the center. The distance of the other point t varies. The blue solid line shows the correlations if both points are on the same side of the line from the center. In particular it can be seen that probability to find two points on this line close to each other is close to zero. If the points

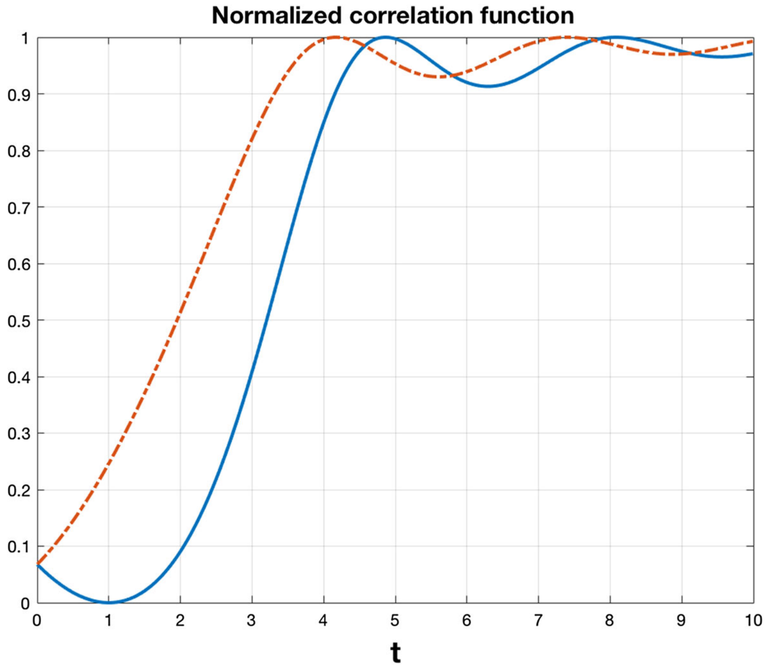

Fig. 2 This shows the plot of $p(us, vt)/(p(us) \times p(vt))$ for $s = 1$ and changing t. The blue solid line is for $\alpha = 0$ (same side of the sphere), and the red dash-dotted line is for $\alpha = \pi$ (opposite side of the sphere) (Color figure online)

are sufficiently far from each other, for example for $s = 1$ and t at least 5 the points become almost uncorrelated.

More precisely, if two points are close to each other, $t \approx s$, then the correlations are quadratic. Namely, if $t = s + h$, then we have

$$p_2^{(c)}(s, s+h) = c_2(s)h^2 + O(h^3),$$

where $c_2(s) = \frac{2}{135\pi} + O(s)$ for small s, and $c_2(s) \sim \frac{2}{3\pi} s^{-4}$ for large s (See Corollary 5.12 for a derivation.).

For comparison, the second correlation function for the classical Ginibre ensemble is

$$p_2^{(G)}(s, s+h) = \frac{1}{\pi^2}\left(1 - e^{-|h|^2}\right) = \frac{1}{\pi^2}|h|^2 + O(|h|^4),$$

for small h, so the correlations are also quadratic. However, the coefficient before $|h|^2$ does not depend on the position s.

The red dash-dotted line shows the situation when the angle between position vectors of the points equals π, that is, when the points are on the opposite sides of the line through the center. In this case the correlation is never zero.

It also turns out that when one of the point is at the center, then the correlation does not depend on the angle between the position vectors of the points. In addition, a calculation shows that if both s and t go to zero, then the second correlation goes to zero as well. This means that the points are still repulsed at the center.

In order to understand the correlations further, let us compute the correlation coefficient between two points that are both located on the sphere of radius s and have angle α between them.

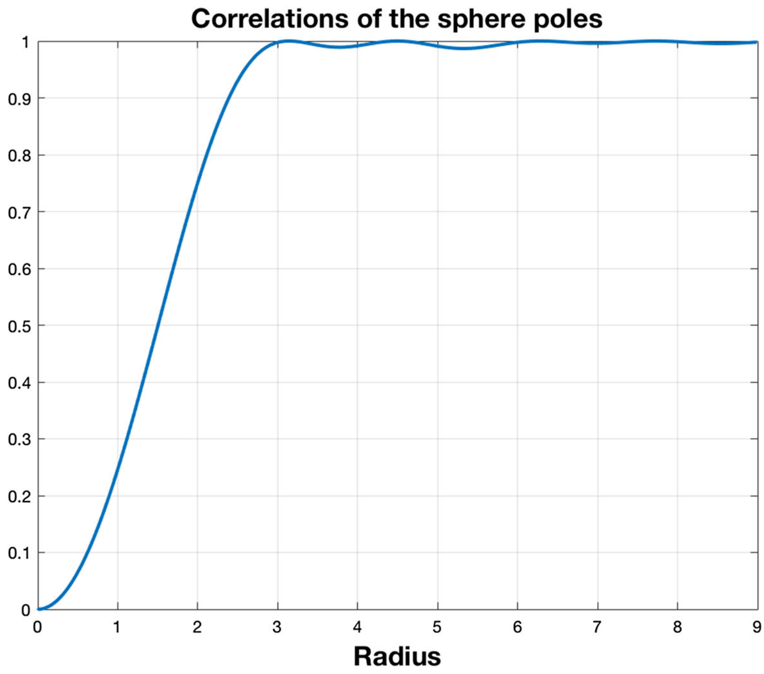

Fig. 3 The dependence of the strength of correlations on radius

From (12),

$$\frac{p^{(2)}(us, vs)}{\rho^{(c)}(us)\rho^{(c)}(vs)} = \frac{1-\cos\alpha}{2}\left\{1-\left(\frac{\frac{\sin(2s)}{2s}-\frac{\sin^2 s}{s^2}}{1-\frac{\sin^2 s}{s^2}}\right)^2\right\}$$

This formula indicate that two points can't be next to each other on the sphere and that the correlations are like in the Poisson field if the points are opposite to each other and the distance from the center is large.

The function

$$f(s) = 1-\left(\frac{\frac{\sin(2s)}{2s}-\frac{\sin^2 s}{s^2}}{1-\frac{\sin^2 s}{s^2}}\right)^2$$

measures how far the correlation deviates from Poisson law for the points which are on opposite poles of the sphere S_s.

The graph of this function is shown in Fig. 3. We can see that the opposite points become essentially pair-wise uncorrelated if the sphere has the radius at least 3.

In the next section we begin a more detailed discussion of these results.

3 Definition of the Quaternion Determinantal Point Field

Let $\Lambda = \mathbb{R}^n$ and the background measure μ be a Radon measure on Λ (that is, a Borel measure which is finite on compact sets).

Definition 3.1 A *random point field* $\mathcal{X}$ on Λ is a random integer-valued positive Radon measure on Λ.

For a set $D \in \Lambda$, we can interpret $\mathcal{X}(D)$ as the number of points in D (counted with multiplicities, perhaps). We will assume our fields to be *simple*, so that each point has a multiplicity at most 1.

A random point field can be described by its correlation functions.

Definition 3.2 A locally integrable function $R_k \colon \Lambda^k \to \mathbb{R}^1_+$ is called a *k-point correlation function* of a random point field $\mathcal{X}$ if for any disjoint measurable subsets $A_1, \ldots, A_m$ of Λ and any non-negative integers $k_1, \ldots, k_m$, such that $\sum_{i=1}^m k_i = k$, the following formula holds:

$$\mathbb{E} \prod_{i=1}^{m} [\mathcal{X}(A_i) \ldots (\mathcal{X}(A_i) - k_i + 1)] \tag{17}$$

$$= \int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} R_k(x_1, \ldots, x_k)\, d\mu(x_1) \cdots d\mu(x_k), \tag{18}$$

where $\mathbb{E}$ denote expectation with respect to random measure $\mathcal{X}$.

If we set all $k_i = 1$ and let A_i to be infinitesimally small, then we can see that $R_k(x_1, \ldots, x_k)$ is the density of the probability to find a point in a neighborhood of each of the points x_i.

More generally, left hand side can be interpreted as a joint ordered moment of the number of points in sets A_i.

A random point field is called determinantal if all of its correlation functions can be represented as determinants:

$$R_k(x_1, \ldots, x_k) = \mathrm{Det}[K(x_i, x_j)]\big|_{1 \le i, j \le k}, \text{ for all } k \in \mathbb{N}, \tag{19}$$

where $K(x, y)$ is a kernel function that maps $\Lambda \times \Lambda$ to $\mathbb{C}$. We extend this definition to include the case of quaternion determinants.

Definition 3.3 A random point field on Λ is *determinantal* if its correlation functions are given by the formula:

$$R_k(x_1, \ldots, x_k) = \mathrm{Det}_{DM}[K(x_i, x_j)]\big|_{1 \le i, j \le k}, \text{ for all } k \in \mathbb{N}, \tag{20}$$

where K maps $\Lambda \times \Lambda$ to $\mathbb{H}$, the skew field of (real) quaternions, and Det_{DM} is the Dyson–Moore determinant.

(The Dyson–Moore determinant is defined for all quaternion matrices A, for which $A_{ij} = \overline{A_{ji}}$, that is, for all self-dual matrices. The eigenvalues of these matrices are real and the determinant is simply the product of eigenvalues (See Appendix A and papers [2,17], and [5] for more information about quaternion matrices and their determinants.).

For the case when kernel $K(x, y)$ takes values in $\mathbb{C}$, this definition agrees with definition in (19).

A basic question is "Which kernels lead to a valid collection of correlation functions?" One sufficient condition is as follows.

Theorem 3.4 (The Dyson–Mehta existence theorem) *Suppose that the kernel $K_N : \Lambda \times \Lambda \to \mathbb{H}$ can be written as*

$$K_N(x, y) = \sum_{k=1}^{N} u_k(x)\, \overline{u}_k(y),$$

Table 1 Scalar products of monomials, $\langle z^m, z^n \rangle$

m/n	0	1	2	3	4	5	6
0	1	0	-3	0	5×3	0	$-7!!$
1	0	3	0	-5×3	0	$7!!$	0
2	-3	0	5×3	0	$-7!!$	0	$9!!$
3	0	-5×3	0	$7!!$	0	$-9!!$	0
4	5×3	0	$-7!!$	0	$9!!$	0	$-11!!$
5	0	$7!!$	0	$-9!!$	0	$11!!$	0
6	$-7!!$	0	$9!!$	0	$-11!!$	0	$13!!$

where $u_k(x)$ is an orthonormal system of quaternion functions on Λ. Then there exists a random point field on Λ with the correlations given by (20), and the total number of points is almost surely equals N.

This result was essentially proved by Dyson [4] and generalized by Mehta [13, Theorem 5.1.4].

By Theorem 3.4 the kernel

$$\mathcal{K}_n(z, w) = \sum_{k=0}^{n-1} \frac{P_k(z)\overline{P_k(w)}}{h_k} \sqrt{f(z) f(w)}$$

defines a valid determinantal field provided that $P_k(z)/\sqrt{h_k}$ are orthonormal polynomials with respect to the measure $d\mu(z)$.

The next section is devoted to the existence and properties of these polynomials. Then we will study the properties of the kernel.

4 Orthogonal Polynomials

4.1 Scalar Product

We look for the orthonormal basis of the space of polynomials $\mathcal{M}$ with respect to the scalar product

$$\langle u(z), w(z)\rangle = \int_\Lambda \overline{u}(z)\, w(z)\, d\mu(z)\,. \tag{21}$$

We can find this basis by applying the Gram-Schmidt procedure to the monomial basis $\{z^k\}\big|_{k=0}^n$. For this purpose we compute the scalar products of monomials. See Table 1.

Recall that $(2k-1)!! := 1 \times 3 \times \cdots \times (2k-3) \times (2k-1)$. (In probability, these numbers occur as moments of Gaussian random variables. If X is a standard Gaussian random variable, then $\mathbb{E}(X^{2k}) = (2k-1)!!$.)

Theorem 4.1 (Scalar products of monomials) *For all non-negative integers m and n*

$$\langle z^m, z^n \rangle = \begin{cases} (-1)^{\frac{n-m}{2}} (m+n+1)!!, & \text{if } n-m \text{ is even,} \\ 0 & \text{if } n-m \text{ is odd.} \end{cases}$$

The proof of this result can be found in Appendix C.

Since all entries in the table of scalar products are integers, we can conclude that the coefficients of all monic orthogonal polynomials are rational.

Table 2 Monic orthogonal polynomials, their squared norms and β_n

n	P_n	h_n	β_n
0	1	1	*
1	z	3	3
2	z^2+3	6	2
3	z^3+5z	30	5
4	z^4+10z^2+15	120	4
5	z^5+14z^3+35z	840	7
6	$z^6+21z^4+105z^2+105$	5040	6
7	$z^7+27z^5+189z^3+315z$	45,360	9
8	$z^8+36z^6+378z^4+1260z^2+945$	362,880	8
9	$z^9+44z^7+594z^5+2772z^3+3465z$	3,991,680	11

4.2 Three-Term Recurrence Relation

By usual means, we can derive the three-term recurrence relation for the quaternion orthogonal polynomials.

Let $P_n(z)$, $n = 0, 1, \ldots$, be a system of monic quaternion polynomials orthogonal with respect to the scalar product in (21). We assume that polynomial $P_n(z)$ has degree n.

Theorem 4.2 (Three-term recurrence for P - polynomials)

1) $P_0(z) = 1$, and $P_1(z) = z$.
2) For $n \geq 1$, polynomials P_n satisfy the following recurrence relation:

$$P_{n+1}(z) = zP_n(z) + \beta_n P_{n-1}(z), \tag{22}$$

where β_n are some real positive coefficients, and

$$\beta_n = \frac{\langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle}.$$

The proof is standard and omitted.

Table 2 is the table of the first orthogonal monic polynomials together with recursion coefficients β_n and the squared norms of polynomials $h_n := \langle P_n, P_n \rangle$. These polynomials are illustrated in Fig. 4.

In the next step, we are going to derive more explicit formulas for the orthogonal polynomials P_n, their squared norms h_n, and coefficients β_n.

4.3 Determinantal Formulas

Let $s_{ij} := \left\langle z^i, z^j \right\rangle$. (These are elements of the infinite matrix in Table 1.) And let D_n denote the principal submatrices of the matrix of scalar products:

$$D_n = \begin{pmatrix} s_{00} & s_{01} & \cdots & s_{0n} \\ s_{10} & s_{11} & \cdots & s_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n0} & s_{n1} & \cdots & s_{nn} \end{pmatrix}.$$

Finally let $|D_n|$ denotes $\det(D_n)$.

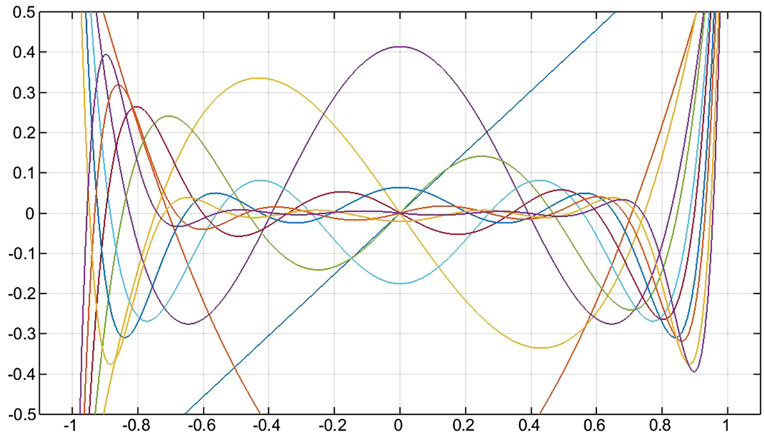

Fig. 4 Plot of $h_n^{-3/4} P_n(i\sqrt{3n}x)$ for $n = 1, \ldots, 9$. The scaling exponent $-3/4$ was chosen ad hoc to fit the plots on the figure

Theorem 4.3 (Determinantal formula for for P-polynomials) *The monic orthogonal polynomials are given by the formula*

$$P_n(z) = \frac{1}{|D_{n-1}|} \det \begin{pmatrix} s_{00} & s_{01} & \cdots & s_{0,n-1} & s_{0n} \\ s_{10} & s_{11} & \cdots & s_{1,n-1} & s_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{n-1,0} & s_{n-1,1} & \cdots & s_{n-1,n-1} & s_{n-1,n} \\ 1 & z & \cdots & z^{n-1} & z^n \end{pmatrix}.$$

Their squared norms are $h_n := \langle P_n(z), P_n(z) \rangle = |D_n| / |D_{n-1}|$.

Proof The polynomials are clearly monic. In order to prove orthogonality, we write

$$\left\langle z^m, P_n(z) \right\rangle = \frac{1}{|D_{n-1}|} \det \begin{pmatrix} s_{00} & s_{01} & \cdots & s_{0,n-1} & s_{0n} \\ s_{10} & s_{11} & \cdots & s_{1,n-1} & s_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{n-1,0} & s_{n-1,1} & \cdots & s_{n-1,n-1} & s_{n-1,n} \\ \langle z^m, 1 \rangle & \langle z^m, z \rangle & \cdots & \left\langle z^m, z^{n-1} \right\rangle & \langle z^m, z^n \rangle \end{pmatrix}.$$

This equals 0 for $m \leq n-1$ because there are two coinciding rows.

For $m = n$, we have $\langle z^n, P_n(z) \rangle = |D_n| / |D_{n-1}|$. Since the polynomials are monic, $\langle P_n(z), P_n(z) \rangle = \langle z^n, P_n(z) \rangle = |D_n| / |D_{n-1}|$. $\square$

4.4 Norm of Polynomials

Theorem 4.4 (Norms and recurrence coefficients for P-polynomials)

$$h_n = \begin{cases} n!(n+2), & \text{if } n \text{ is odd,} \\ (n+1)!, & \text{if } n \text{ is even.} \end{cases}$$

$$\beta_n = \begin{cases} n+2, & \text{if } n \text{ is odd,} \\ n, & \text{if } n \text{ is even.} \end{cases}$$

Table 3 $Q_n(x)$ polynomials

n	$Q_n(x)$
0	1
1	x
2	$x^2 - 3$
3	$x^3 - 5x$
4	$x^4 - 10x^2 + 15$
5	$x^5 - 14x^3 + 35x$
6	$x^6 - 21x^4 + 105x^2 - 105$
7	$x^7 - 27x^5 + 189x^3 - 315x$
8	$x^8 - 36x^6 + 378x^4 - 1260x^2 + 945$
9	$x^9 - 44x^7 + 594x^5 - 2772x^3 + 3465x$

The proof proceeds by exhibiting an explicit formula for the determinant $|D_n|$. Since the proof is rather lengthy, we omit it due to the space constraints.[4]

4.5 Q Polynomials

Let s be real and define the polynomials $Q(s)$ by the formula:

$$Q(s) = \mathbf{i}^{-n} P_n(\mathbf{i}s). \tag{23}$$

The first ten Q_n are shown in Table 3.

Theorem 4.5 (Properties of Q-polynomials)

(i) Polynomials $Q_n(x)$ satisfy the following recursion:

$$Q_{n+1}(x) = x Q_n(x) - \beta_n Q_{n-1}(x), \tag{24}$$

(ii) Polynomials $Q_n(x)$ are orthogonal with respect to a non-negative measure ν on $\mathbb{R}$.
(iii) The coefficients of every polynomial $Q_n(x)$ are real.
(iv) All the zeros of a polynomial $Q_n(x)$ are simple and real.
(v) Any two zeros of a polynomial $Q_n(x)$ are separated by a zero of polynomial $Q_{n-1}(x)$ and vice versa.

Proof Formula (24) follows from the properties of $P_n(x)$. Claim (ii) follows by Favard's theorem, because β_n are positive. Claim (iii) is implied by (i) becase β_n are real. Claims (iv) and (v) are implied by (ii), see Theorem 1.2.2 in Akhieser [1]. □

Theorem 4.6 (Orthogonality of Q-polynomials) *The polynomials $Q_n(x)$ are monic orthogonal polynomials with respect to measure ν with density*

$$f(t) = \frac{1}{\sqrt{2\pi}} t^2 e^{-t^2/2},$$

defined on all real line.

[4] The proof can be found at the author's webpage ("Notes on the properties of quaternion orthogonal polynomials").

Proof The moments of this measure are $m_{2k+1} = 0$ and $m_{2k} = (2k+1)!!$. By using these moments, we can calculate the coefficients in the 3-term recurrence relation for orthogonal polynomials related to this measure. It turns out that these are the same coefficients as that for the polynomials Q_n. Since the initial conditions are also satisfied, Q_n are the monic orthogonal polynomials for the measure ν. □

Theorem 4.7 (Relation of the Q and Laguerre polynomials)

$$Q_{2n}(x) = (-2)^n n!\, L_n^{(1/2)}\left(\frac{x^2}{2}\right),$$

$$Q_{2n+1}(x) = (-2)^n n!\, x\, L_n^{(3/2)}\left(\frac{x^2}{2}\right),$$

where $L_n^{(\alpha)}(x)$ *denote the Laguerre polynomials.*

Proof The orthogonality relations for the Laguerre polynomials are

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = \delta_{n\,m}\Gamma(\alpha+1)\binom{n+\alpha}{n}.$$

After a change of variable in this relations, the claim of the theorem directly follows from Theorem 4.6. □

5 Kernel

5.1 An Explicit Expression for the Kernel

Recall that the kernel is defined as

$$K_n(x, y) = \sum_{k=0}^{n} \frac{P_k(x)\,\overline{P_k(y)}}{h_k}$$

It turns out that the kernel admits a simpler expression which avoids a long summation. As in the classical case, this expression is based on the availability of the 3-term recurrence relation for the orthogonal polynomials, and it can be called an analogue of the Christoffel–Darboux relation.

Theorem 5.1 (The Christoffel–Darboux formula) *The following formula holds for all* $n \geq 0$, *and all imaginary quaternions* x *and* y*:*

$$x K_n(x, y) + K_n(x, y)\,\overline{y} = \frac{P_{n+1}(x)\,\overline{P_n(y)} + P_n(x)\,\overline{P_{n+1}(y)}}{h_n}. \tag{25}$$

Proof By using the three-term recurrence relation (22) and setting $P_{-1}(x) = 0$, we can write

$$x K_n(x, y) = \sum_{k=0}^{n} \frac{(P_{k+1}(x) - \beta_k P_{k-1}(x))\overline{P_k(y)}}{h_k}.$$

Similarly,

$$K_n(x, y)\,\overline{y} = \sum_{k=0}^{n} \frac{P_k(x)\,\overline{y P_k(y)}}{h_k} = \sum_{k=0}^{n} \frac{P_k(x)\left(\overline{P_{k+1}(y)} - \beta_k \overline{P_{k-1}(y)}\right)}{h_k}.$$

If we add these two expressions together and use the fact that $\beta_k = h_k/h_{k-1}$, then we obtain Eq. (25). □

We find the kernel from the relation (25) by using a result from Janovská and Opfer [9].

An equation $Az + zB = C$ is called *singular* if the equation $Az + zB = 0$ has a non-zero solution $z \in \mathbb{H}$.

Lemma 5.2 (The Janovska–Opfer solution method) *The equation*

$$Az + zB = C; \quad A, B, C, z \in \mathbb{H},\ AB \neq 0,$$

is singular if and only if A and $-B$ are equivalent (that is, $|A| = |B|$, and $\mathrm{Re}(A) = \mathrm{Re}(-B)$*). If it is non-singular, then its solution is*

$$z = f_l^{-1}\left(C + A^{-1}C\overline{B}\right) = (C + \overline{A}CB^{-1})f_r^{-1}, \tag{26}$$

where $f_l = 2\mathrm{Re}\,(B) + A + |B|^2 A^{-1}$, *and* $f_r = 2\mathrm{Re}(A) + B + |A|^2 B^{-1}$.

Sketch of the proof: Assume that A and B are not equivalent. This implies that f_l and f_r are not zero. Then, for the first equality in (26), we have

$$\begin{aligned} C + A^{-1}C\overline{B} &= Az + zB + A^{-1}\big(Ax + xB\big)\overline{B} \\ &= z\big(B + \overline{B}\big) + Az + A^{-1}z\big|B\big|^2 \\ &= \big(2\mathrm{Re}(B) + A + \big|B\big|^2 A^{-1}\big)z. \end{aligned}$$

Hence $z = f_l^{-1}\big(C + A^{-1}C\overline{B}\big)$. The second equality is proved in a similar way. □

Theorem 5.3 (Explicit formula for the kernel) *Let $x = us$ and $y = vt$, where $s = |x|$ and $t = |y|$. Suppose $s \neq t$. Then*

$$K_n(us, vt) = \rho_n(s,t)\frac{1 - uv}{2} + (-1)^n \delta_n(s,t)\frac{1 + uv}{2},$$

where

$$\begin{aligned} \rho_n(s,t) &= \frac{Q_n(s)Q_{n+1}(t) - Q_{n+1}(s)Q_n(t)}{h_n(t - s)}, \quad \textit{and} \\ \delta_n(s,t) &= \frac{Q_n(s)Q_{n+1}(t) + Q_{n+1}(s)Q_n(t)}{h_n(t + s)} \end{aligned}$$

and $Q_n(s) = \mathbf{i}^{-n} P_n(\mathbf{i}s)$.

Proof We apply Theorem 5.2 with $z = K_n(x, y)$, $A = x$, $B = \overline{y}$, and $C_n = h_n^{-1}(P_{n+1}(x)\overline{P_n(y)} + P_n(x)\overline{P_{n+1}(y)})$.

Using the fact that x and y are purely imaginary, and therefore $\mathrm{Re}(x) = \mathrm{Re}(y) = 0$, $\overline{x} = -x$, $\overline{y} = -y$, we calculate

$$f_l = x + |\overline{y}|^2 x^{-1}, \quad f_l^{-1} = \frac{x}{x^2 - y^2},$$

$$K_n(x, y) = \frac{1}{x^2 - y^2}(xC_n + C_n y).$$

Next, we substitute the expression for C_n and obtain the equation

$$K_n(x, y) = \frac{1}{h_n(x^2 - y^2)} \Big(x \Big[P_{n+1}(x)\overline{P_n(y)} + P_n(x)\overline{P_{n+1}(y)} \Big] \\ + \Big[P_{n+1}(x)\overline{P_n(y)} + P_n(x)\overline{P_{n+1}(y)} \Big] y \Big).$$

Next, let $x = us$ and $y = vt$, where $s = |x|$ and $t = |y|$.
Then, after some calculations we find that for odd n

$$K_n(x, y) = \frac{1}{h_n(t^2 - s^2)} \Big(Q_n(s) Q_{n+1}(t) \big(s - uvt \big) - Q_{n+1}(s) Q_n(t) \big(t - uvs \big) \Big).$$

For even n,

$$K_n(x, y) = \frac{1}{h_n(t^2 - s^2)} \Big(Q_n(s) Q_{n+1}(t) \big(t - uvs \big) - Q_{n+1}(s) Q_n(t) \big(s - uvt \big) \Big).$$

The statement of the theorem follows after a term rearrangement. □

5.2 Rotational Invariance of the Kernel

Define the following action of $\mathbb{H} \setminus \{0\}$ on Λ.

$$\mathrm{Ad}_q x = q x q^{-1}.$$

Recall that this action represents three-dimensional rotations. Namely, if $x = x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k}$, and q is the unit quaternion $q = \cos\theta + (q_1\mathbf{i}+q_2\mathbf{i}+q_3\mathbf{i})\sin\theta$, then $\mathrm{Ad}_q x = y_1\mathbf{i}+y_2\mathbf{j}+y_3\mathbf{k}$, and the vector (y_1, y_2, y_3) is the image of the vector (x_1, x_2, x_3) after a rotation around the axis with direction vector (q_1, q_2, q_3) by the angle θ.

Corollary 5.4 *Let x and y be imaginary quaternions. Then the following rotational invariance holds:*

$$K_n(\mathrm{Ad}_q x, \mathrm{Ad}_q y) = K_n(x, y).$$

Proof This follows directly from Theorem 5.3. □

5.3 Asymptotic Behavior of the Kernel in the Bulk

What happens when the number of particles n grows?

It turns out that almost all particles are contained in the ball of the radius $2\sqrt{n}$, and it is convenient to introduce new variables φ, ψ in the following way,

$$\varphi = \arccos\left(\tfrac{s}{2\sqrt{n+3/2}}\right),$$
$$\psi = \arccos\left(\tfrac{t}{2\sqrt{n+3/2}}\right).$$

In addition, let us introduce the following notation:

$$\overline{x|} := \sin(2x) - 2x = -\frac{1}{6}(2x)^3 + O(x^5),$$
$$\text{Ж}_n(\varphi, \psi) := \frac{n+3/2}{2}(\overline{\varphi|} - \overline{\psi|}),$$
$$\text{Ш}_n(\varphi, \psi) = \frac{n+3/2}{2}(\overline{\varphi|} + \overline{\psi|}). \tag{27}$$

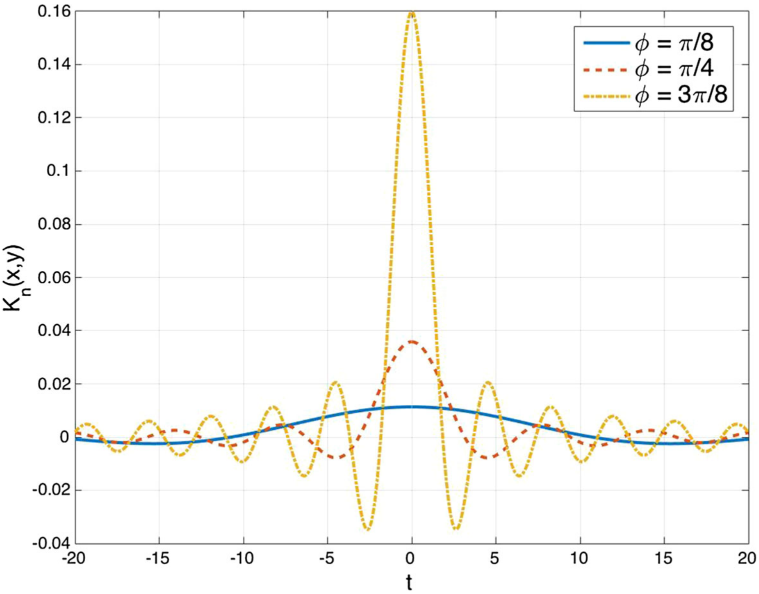

Fig. 5 A plot of $\lim_{n\to\infty} \hat{K}_n(x, x + \frac{t}{n})$ for various values of parameters $\varphi = \arccos x$ and t

Due to the formula for the kernel in Theorem 5.3, it is enough to consider the asymptotic behavior of functions $\rho_n(s,t)$ and $\delta_n(s,t)$.

Theorem 5.5 (Asymptotic expressions for ρ_n and δ_n) *Let $s, t \in [\varepsilon\sqrt{n}, (2-\varepsilon)\sqrt{n}]$ and $s \neq t$. Then,*

$$\begin{aligned}\rho_n(s,t)e^{-\frac{s^2+t^2}{4}} &= \sqrt{\frac{2}{\pi}\frac{1}{8}\Big(n+\frac{3}{2}\Big)^{-3/2}} \\ &\quad\times\frac{1}{(\cos\varphi-\cos\psi)\cos\varphi\cos\psi\sqrt{\sin\varphi\sin\psi}} \\ &\quad\times\frac{1}{2}\Big\{\cos(\mathcal{K}_n(\varphi,\psi)-\varphi)-\cos(\mathcal{K}_n(\varphi,\psi)+\psi) \\ &\quad-\sin(\text{Ш}_n(\varphi,\psi)-\psi)+\sin(\text{Ш}_n(\varphi,\psi)-\varphi)+O_\varepsilon\Big(\frac{1}{\sqrt{n}}\Big)\Big\}. \end{aligned} \tag{28}$$

and

$$\delta_n(s,t)e^{-\frac{s^2+t^2}{4}} = O_\varepsilon\Big(\frac{1}{n}\Big).$$

Proof is based on the Plancherel–Rotach approximation and is relegated to Appendix D. Let

$$\begin{aligned} &x_0 \in [\varepsilon, 1-\varepsilon],\ \xi = \arccos x_0 \in (0, \pi/2),\ \text{ and } k = (2\sin^2\xi)^{-1}, \\ &s_n = 2\sqrt{n+3/2}\cos\Big(\xi+\frac{k\sigma}{n}\Big), \\ &t_n = 2\sqrt{n+3/2}\cos\Big(\xi+\frac{k\tau}{n}\Big), \end{aligned}$$

Theorem 5.6 (Scaling limit for ρ_n) *For $x_0 \in [\varepsilon, 1-\varepsilon]$, and s_n, t_n as defined above, we have*

$$2\sqrt{n+\frac{3}{2}}\rho_n(s_n,t_n)\sqrt{f(s_n)f(t_n)} = \frac{\sin(\tau-\sigma)}{\tau-\sigma}\rho(x_0) + O_\varepsilon(n^{-1/2}).$$

Here $f(s)$ is the density of the background measure as in (8) and $\rho(s)$ is the density of the limit measure as in (13). The scaling limit for various values of parameters is illustrated in Fig. 5.

Proof We use

$$\varphi = \xi + \frac{k\sigma}{n},$$
$$\psi = \xi + \frac{k\tau}{n},$$

in Theorem 5.5. Then, for the last two terms in (28), we have:

$$\begin{aligned}\sin\left(\text{Ш}_n - \xi - \frac{k\tau}{n}\right) - \sin\left(\text{Ш}_n - \xi - \frac{k\sigma}{n}\right) &= \sin(\text{Ш}_n - \xi)\left(\cos\frac{k\tau}{n} - \cos\frac{k\sigma}{n}\right)\\ &\quad - \cos(\text{Ш}_n - \xi)\left(\sin\frac{k\tau}{n} - \sin\frac{k\sigma}{n}\right)\\ &= O_\varepsilon\left(n^{-1}\right).\end{aligned}$$

For the first two terms, we use the fact that

$$\begin{aligned}\text{Ж}_n\left(\xi + \frac{k\sigma}{n}, \xi + \frac{k\tau}{n}\right) &\equiv \frac{n+\frac{3}{2}}{2}\left(\overline{\varphi + \frac{k\sigma}{n}}\Big| - \overline{\varphi + \frac{k\tau}{n}}\Big|\right)\\ &\equiv \frac{n+\frac{3}{2}}{2}\left[\sin\left(2\xi + \frac{2k\sigma}{n}\right) - \sin\left(2\xi + \frac{2k\tau}{n}\right)\right.\\ &\quad \left. + \frac{2k(\tau-\sigma)}{n}\right]\\ &= (1-\cos 2\xi)k(\tau-\sigma) + O_\varepsilon\left(n^{-1}\right)\\ &= \tau - \sigma + O_\varepsilon\left(n^{-1}\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\cos\left(\text{Ж}_n - \xi - \frac{k\sigma}{n}\right) - \cos\left(\text{Ж}_n + \xi + \frac{k\tau}{n}\right)\\ &= \cos(\tau - \sigma - \xi) - \cos(\tau - \sigma + \xi) + O_\varepsilon(n^{-1})\\ &= 2\sin(\tau-\sigma)\sin\xi + O_\varepsilon(n^{-1}).\end{aligned}$$

Therefore, after some calculation we get from Theorem 5.5,

$$2\sqrt{n+3/2}\rho_n(s_n,t_n)\left(\frac{1}{2\pi}\right)^{3/2}e^{-\frac{x_n^2+y_n^2}{4}} = \frac{1}{(2\pi)^2}\frac{\sin(\tau-\sigma)}{\tau-\sigma}\frac{\sin\xi}{\cos^2\xi} + O_\varepsilon(n^{-1/2}).$$

Since

$$\frac{1}{(2\pi)^2}\frac{\sin\xi}{\cos^2\xi} = \frac{1}{(2\pi)^2}\frac{\sqrt{1-x_0^2}}{x_0^2} = \rho(x_0),$$

the claim of the theorem follows. $\square$

Theorem 5.7 (Limit density) *Let $\rho_n(x)$ denote the density of the quaternion Ginibre field at point $x \in \mathbb{R}^3$ with respect to the Lebesque measure on $\mathbb{R}^3$. Assume that u is a pure unit quaternion and $s \in \mathbb{R}$, $s > 0$. Then,*

$$\lim_{n\to\infty} 2\sqrt{n}\rho_n\big(2\sqrt{n}us\big) = \begin{cases} \rho(s), & \text{if } s < 1, \\ 0 & \text{if } s > 1. \end{cases}$$

Proof This theorem immediately follows from Theorem 5.6 if we take σ and τ to zero. □

Corollary 5.8 (Limit radial density) *For the limit of radial density, we have*

$$\lim_{n\to\infty} \frac{1}{2\sqrt{n}}\mathfrak{p}_n\big(2\sqrt{ns}\big) = \begin{cases} \mathfrak{p}(s) := \frac{1}{\pi}\sqrt{1-s^2}, & \text{if } 0 < s < 1, \\ 0 & \text{if } s > 1, \end{cases}$$

Proof This result follows from the definition of the radial density by a calculation. □

Theorem 5.9 (Scaling limit of the kernel in the bulk) *Let $\sigma, \tau \in \mathbb{R}$ and $u, v \in S^2$, the space of pure unit quaternions. Then,*

$$\lim_{n\to\infty} \mathcal{K}_n\big((u,\sigma),(v,\tau);x_0\big) = \mathcal{K}\big((u,\sigma),(v,\tau)\big) := \frac{\sin(\tau-\sigma)}{\tau-\sigma}\frac{1-uv}{2}.$$

Proof From Theorem 5.5 (asymptotic expressions for δ_n and ρ_n), we see that $\sqrt{n}\delta_n(s_n, t_n) = O\big(n^{-1/2}\big)$. Then, the claim of the theorem follows by combination of results in Theorems 5.3 (an explicit formula for the kernel) and 5.6 (scaling limit for ρ_n). □

5.4 Asymptotic Behavior of the Kernel at the Origin

Let z and z' be pure quaternions. We define the limit kernel at the center as follows. Suppose $s \neq t$. Then

$$K^{(c)}(z,z') := \lim_{n\to\infty}\left(n+\frac{3}{2}\right)^{-3/2} \mathcal{K}_n\left(\tfrac{z}{\sqrt{n+3/2}}, \tfrac{z'}{\sqrt{n+3/2}}\right)$$

Theorem 5.10 (Scaling limit of the kernel at the origin) *Let z and z' be pure quaternions, $z = us$ and $z' = vt$, where $s = |z|$ and $t = |z'|$. Then the limit $K^{(c)}(z,z')$ exists and*

$$\begin{aligned} K^{(c)}(us,vt) = \frac{1}{\sqrt{2\pi}}\frac{1}{st}\Bigg\{&\left[\frac{\sin(t-s)}{t-s} - \frac{\sin(t+s)}{t+s}\right] \\ &-uv\left[\frac{\sin(t-s)}{t-s} + \frac{\sin(t+s)}{t+s} - 2\frac{\sin t}{t}\times\frac{\sin s}{s}\right]\Bigg\}, \end{aligned}$$

Proof In order to study the situation at the origin of the coordinates, we use Theorems 5.3 (an explicit formula for the kernel), 4.7 (relation of the Q and Laguerre polynomials), and the asymptotics from Theorem B.2 (the Hilb approximation). Then we get

$$\frac{1}{\sqrt{h_n}}Q_n(s)e^{-s^2/4} = (-1)^{\lfloor n/2\rfloor}\left(\tfrac{\pi}{2}\right)^{1/4}\tfrac{1}{\sqrt{s}}J_\alpha\Big(\sqrt{n+3/2}\,s\Big) + \tfrac{1}{\sqrt{s}}O\Big((n!)^{-1/2}\Big),$$

where $\alpha = 1/2$ if n is even and $\alpha = 3/2$ if n is odd.

This leads to asymptotic expressions for the components of the kernel.

Let $x = \sqrt{n+3/2}\,s > 0$ and $y = \sqrt{n+3/2}\,t > 0$. Then,

$$\left(n+\frac{3}{2}\right)^{-3/2} \times \rho_n\left(\frac{x}{\sqrt{n+3/2}}\right) = \sqrt{\frac{\pi}{2}}\frac{J_{1/2}(x)J_{3/2}(y) - J_{3/2}(x)J_{1/2}(y)}{(y-x)\sqrt{xy}} + \frac{O\left((n!)^{-1/2}\right)}{(y-x)\sqrt{xy}}$$

and

$$\left(n+\frac{3}{2}\right)^{-3/2} \times (-1)^n\delta_n\left(\frac{x}{\sqrt{n+3/2}}\right) = \sqrt{\frac{\pi}{2}}\frac{J_{1/2}(x)J_{3/2}(y) + J_{3/2}(x)J_{1/2}(y)}{(x+y)\sqrt{xy}} + \frac{O\left((n!)^{-1/2}\right)}{(x+y)\sqrt{xy}}$$

Hence

$$K^{(c)}(us, vt) = \frac{\rho(s,t)+\delta(s,t)}{2} - uv\,\frac{\rho(s,t)-\delta(s,t)}{2},$$

where

$$\rho(s,t) = \sqrt{\frac{\pi}{2}}\frac{J_{1/2}(s)J_{3/2}(t) - J_{3/2}(s)J_{1/2}(t)}{(t-s)\sqrt{st}}, \text{ and}$$
$$\delta(s,t) = \sqrt{\frac{\pi}{2}}\frac{J_{1/2}(s)J_{3/2}(t) + J_{3/2}(s)J_{1/2}(t)}{(t+s)\sqrt{st}}.$$

By using the identities

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$
$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x^3}}(\sin x - x\cos x),$$

we can simplify the expressions for $\rho(s,t)$ and $\delta(s,t)$ to

$$\rho(s,t) = \sqrt{\frac{2}{\pi}}\frac{1}{st}\left[\frac{\sin(t-s)}{t-s} - \frac{\sin t}{t}\times\frac{\sin s}{s}\right], \text{ and} \tag{29}$$
$$\delta(s,t) = \sqrt{\frac{2}{\pi}}\frac{1}{st}\left[\frac{\sin t}{t}\times\frac{\sin s}{s} - \frac{\sin(t+s)}{t+s}\right]. \tag{30}$$

Hence, the kernel is

$$K^{(c)}(us, vt) = \frac{1}{\sqrt{2\pi}}\frac{1}{st}\left\{\left[\frac{\sin(t-s)}{t-s} - \frac{\sin(t+s)}{t+s}\right] - uv\left[\frac{\sin(t-s)}{t-s} + \frac{\sin(t+s)}{t+s} - 2\frac{\sin t}{t}\times\frac{\sin s}{s}\right]\right\}.$$

□

From this theorem we can get expressions for the first and the second correlation functions.

Corollary 5.11

$$p_1^{(c)}(us) \equiv \lim_{t\to s} K^{(c)}(us, ut) = \sqrt{\frac{2}{\pi}}\frac{1}{s^2}\left[1 - \left(\frac{\sin s}{s}\right)^2\right].$$

Corollary 5.12

$$p_2^{(c)}(us, vt) = p_1^{(c)}(s)p_1^{(c)}(t) - \frac{1}{2}\Big(A(s,t) + B(s,t)\cos\alpha\Big),$$

where

$$\begin{aligned} A(s,t) = \frac{2}{\pi}\frac{1}{s^2t^2}\Bigg\{ & \left(\frac{\sin(t-s)}{t-s}\right)^2 + \left(\frac{\sin(t+s)}{t+s}\right)^2 + 2\frac{\sin^2 s}{s^2}\frac{\sin^2 t}{t^2} \\ & -2\frac{\sin s}{s}\frac{\sin t}{t}\left[\frac{\sin(t-s)}{t-s} + \frac{\sin(t+s)}{t+s}\right]\Bigg\} \end{aligned}$$

and

$$\begin{aligned} B(s,t) = \frac{2}{\pi}\frac{1}{s^2t^2}\Bigg\{ & \left(\frac{\sin(t-s)}{t-s}\right)^2 - \left(\frac{\sin(t+s)}{t+s}\right)^2 \\ & -2\frac{\sin s}{s}\frac{\sin t}{t}\left[\frac{\sin(t-s)}{t-s} - \frac{\sin(t+s)}{t+s}\right]\Bigg\}. \end{aligned}$$

For $t = s + h$, we can calculate the Taylor expansion of the second correlation in terms of h. This gives

$$p_2^{(c)}(s, s+h) = c_2(s)h^2 + O(h^3),$$

where

$$c_2(s) = \frac{2}{3\pi s^8} \times \Big(s^4 - 3s^2\cos^2(s) + 6s\sin(s)\cos(s) - (s^2+3)\sin^2(s)\Big).$$

For small s this gives $c_2(s) = \frac{2}{135\pi} + O(s)$.

6 Summary and Open Questions

We introduced a new family of determinantal fields in the 3-dimensional real space. This field closely connected to certain orthogonal polynomials where orthogonality is defined over the right module of quaternion functions. We derived an explicit formula for kernels of these determinantal fields and computed their scaling limit in the bulk of the point distribution and at the center.

Several problems seems to be interesting to investigate:

(1) What is the distribution of the hole sizes in the point field? If we select a point $x \in \mathbb{R}^3$, then what is the distribution of its closest field point?
(2) It would be interesting to see if this field can be realized as a field of eigenvalues of some quaternion matrices.
(3) Is there a random dynamical system, for which the field distribution is an equilibrium distribution? Here we have in mind a system which would resemble the Dyson Brownian motion.
(4) Is it possible to adapt the constructions in this paper to define a random point field in $\mathbb{R}^4$?

Appendix A: Background on Quaternion Matrices and Determinants

The $\mathbb{R}$-algebra of quaternions $\mathbb{H}$ is generated by elements $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, which are anticommutative and satisfy the multiplication rules: $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, and $\mathbf{ki} = \mathbf{j}$. This algebra is a non-commutative division algebra, meaning that every non-zero element has a unique multiplicative inverse. The field $\mathbb{C}$ can be realized as a real subalgebra of $\mathbb{H}$: $a_1 + a_2 i \to a_1\mathbf{e} + a_2\mathbf{i}$, however, this subalgebra does not belong to the center of $\mathbb{H}$, and cannot be used to make $\mathbb{H}$ into a $\mathbb{C}$-algebra.

A useful representation of $\mathbb{H}$ is as a real subalgebra of the $\mathbb{C}$-algebra $\mathbb{M}_2(\mathbb{C})$. For $z = a_1\mathbf{e} + a_2\mathbf{i}+a_3\mathbf{j}+a_4\mathbf{k}$, let $x = a_1 + a_2 i$ and $y = a_3 + a_4 i$ be two complex numbers. Then z can be represented by:

$$\varphi(z) = \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix},$$

and this representation respect the addition and multiplication operations in $\mathbb{H}$.

The conjugate of a quaternion $z = a_1\mathbf{e} + a_2\mathbf{i}+a_3\mathbf{j}+a_4\mathbf{k}$ is defined as $\overline{z} = a_1\mathbf{e} - a_2\mathbf{i}-a_3\mathbf{j}-a_4\mathbf{k}$. This is an involution and an anti-isomorphism of $\mathbb{H}$, since $\overline{\overline{z}} = z$ and $\overline{zw} = (\overline{w})(\overline{z})$. The norm of z is defined as $|z| = \sqrt{z\overline{z}} > 0$, and it is indeed a norm in the sense of C-algebras. (In the matrix representation, $\varphi(\overline{z}) = \varphi(z)^*$, where $(A^*)_{ij} := \overline{A}_{ji}$, and $|z|^2 = \frac{1}{2}\text{trace}\big[\varphi(z)^*\varphi(z)\big]$.)

A quaternion matrix X is a matrix with quaternion entries. Its dual matrix X^* has the entries $(X^*)_{ij} = \overline{X}_{ji}$. Then $(XY)^* = Y^*X^*$. The matrix is called unitary if $XX^* = I$ and self-dual (or Hermitian) if $X^* = X$.

The map φ extends to quaternion matrices so that a matrix in $M_n(\mathbb{H})$ is represented by a matrix in $M_{2n}(\mathbb{C})$

Define $J = \varphi(\mathbf{j}) \otimes I_n$. That is, J is a $2n$ -by-$2n$ block-diagonal matrix with the blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the main diagonal. Then, a quaternion matrix X is self-dual if and only if $J\varphi(x)$ is anti-symmetric, $[J\varphi(X)]^T = -J\varphi(X)$.

A quaternion λ is called a (right) eigenvalue of a quaternion matrix X if for some non-zero quaternion vector v, we have $Xv = v\lambda$. It is easy to see that if λ is an eigenvalue, then $q^{-1}\lambda q$ is also an eigenvalue for any quaternion q, with eigenvector vq, which implies that typically a quaternion matrix will have an infinity of eigenvalues. However, for self-dual quaternion matrices it is possible to show that all eigenvalues are real and that every n-by-n self-dual matrix X has exactly n eigenvalues (counting with multiplicities).

Moreover, the spectral theorem holds. Namely, for every self-dual quaternion matrix X, there exists a real diagonal matrix D and a unitary quaternion matrix U, such that $X = U^*DU$. This result goes back to Teichmuller. For a modern account, see Farenick and Pidkowich and also see references in Zhang [17].

It is possible to generalize the concept of determinant to quaternion matrices. There are several sensible ways to do this and interested reader can find details in a review paper by Aslaksen [2]. We use the Moore-Dyson determinant [4]. For a self-dual matrix X, the determinant $\text{Det}_M(X)$ can be defined as a product of the eigenvalues of X. In particular, the determinant is real, and it is non-negative for positive-definite self-dual matrices.

Remarkably, this determinant can also be defined by using a variant of the Cayley combinatorial formula for the determinant. Namely, let S_n be the group of permutations of the set $\{1, \ldots, n\}$. Write every permutation σ as a product of cycles:

$$\sigma = (n_1 i_2 \ldots i_s)(n_2 j_2 \ldots j_t) \ldots (n_r k_2 \ldots k_l),$$

where n_i are the largest elements of each cycle and $n_1 > n_2 > \cdots > n_r$. Then we can write

$$\mathrm{Det}_M\,(X) = \sum_{\sigma} \varepsilon\,(\sigma)\left(X_{n_1 i_2} X_{i_2 i_3} \ldots X_{i_s n_1}\right) \ldots \left(X_{n_r k_2} X_{k_2 k_3} \ldots X_{k_l n_r}\right),$$

where $\varepsilon\,(\sigma) = (-1)^{n-r}$ is the sign of the permutation σ.

The Moore-Dyson determinant can also be computed in terms of the matrix representation $\varphi(X)$ as a Pfaffian of a related matrix,

$$\mathrm{Det}_M\,(X) = \mathrm{Pf}\,(-J\varphi\,(X))\,. \tag{31}$$

(See [4] and Proposition 6.1.5 on p. 238 in Forrester's book [6].) By using properties of the Pfaffian, one has

$$\mathrm{Det}_M\left(X^* X\right) = \det\,(\varphi\,(X))\,. \tag{32}$$

(See Formula (6.13) on page 239 in [6] or Corollary 5.1.3 on p. 75 in [13].)

Appendix B: Useful Asymptotic Formulas

The following is a modification of the standard Plancherel–Rotach for the version of Hermite polynomials that we use in this paper. (See Theorem 8.22.9 on p. 201 in [15] for the standard version.)

Theorem B.1 (Plancherel–Rotach approximation) *Let ϵ and ω be fixed positive numbers. For $x = 2\sqrt{n+\frac{1}{2}}\cos\varphi$, $\epsilon \leq \varphi \leq \pi - \epsilon$, we have*

$$\begin{aligned} e^{-x^2/4} H_n(x) =& \left(\frac{2}{\pi n}\right)^{\frac{1}{4}} \sqrt{n!}\,\frac{1}{\sqrt{\sin\varphi}} \\ &\cdot \left\{ \sin\left[\left(\frac{2n+1}{4}\right)\left(\sin\,2\varphi - 2\varphi\right) + \frac{3\pi}{4}\right] + O(n^{-1})\right\}. \end{aligned}$$

The second approximation will be used for the asymptotics at the origin of the coordinates. Let $N = n + (\alpha+1)/2$.

Theorem B.2 (Hilb approximation)

$$e^{-x/2} x^{\alpha/2} L_n^{(\alpha)}(x) = N^{-\alpha/2}\tfrac{\Gamma(n+\alpha+1)}{n!} J_\alpha\left(2\sqrt{Nx}\right) + O(n^{\alpha/2-3/4}),$$

uniformly in $0 < x \leq \omega$.

More precisely, the error term is $x^{5/4} O\left(n^{\alpha/2-3/4}\right)$, if $cn^{-1} \leq x \leq \omega$, and $x^{\alpha/2+2} O(n^{\alpha})$, if $0 < x \leq cn^{-1}$.

(See Theorem 8.22.4 on p. 199 in [15].)

Appendix C: Proof of Theorem 4.1

We start with a useful lemma.

Lemma C.1 *For every non-negative integers l and k, the integral $\int_\Lambda |z|^{2l}\, z^k d\mu\,(z)$ is real.*

Proof We write $z = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and expand the expression $|z|^{2l} z^k$. We claim that every monomial coefficient before $\mathbf{i}$, $\mathbf{j}$, or $\mathbf{k}$ in this expansion has one of its variables x_i in the odd power. If this claim holds, then the integral of these monomials with respect to measure μ is 0, by the symmetry of μ, and the lemma is proved.

It is sufficient to prove the claim for $l = 0$, since $|z|^{2l} = \left(x_1^2 + x_2^2 + x_3^2\right)^l$ is real and all monomials in its expansion have variables in the even power.

Consider a single term in the expansion of z^k, for example, $x_1\mathbf{i}x_3\mathbf{k}x_2\mathbf{j}x_2\mathbf{j}\ldots$, It can be either imaginary or real, and it is clear that it is imaginary if and only if the term contain at least one of the variables in the odd power. Indeed, we can do transpositions of imaginary units in the expansion and this will only introduce real factors. Hence, if all powers are even then all imaginary units in the product can be paired off and cancelled out, so that the product is real.

Therefore the claim and the lemma are proved. □

Now, let us calculate the real part of the expression $|z|^{2l} z^k$. Since $|z|^{2l}$ is real, we only need to calculate the real part of z^k.

Lemma C.2 *Let* $z = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$. *Then,*

$$\operatorname{Re} z^k = \operatorname{Re} \overline{z}^k = \begin{cases} (-1)^r \left(x_1^2 + x_2^2 + x_3^2\right)^r, & \textit{if } k = 2r, \\ 0, & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof We write the quaternion z in its matrix form:

$$\varphi(z) = \begin{pmatrix} x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & -x_1 i \end{pmatrix},$$

and note that for every quaternion w its real part can be computed as $\frac{1}{2}\operatorname{Tr}\varphi(w)$. The eigenvalues of $\varphi(z)$ are $\pm i\sqrt{x_1^2 + x_2^2 + x_3^2}$. Hence, we compute:

$$\operatorname{Re} z^k = \frac{1}{2}\operatorname{Tr}\varphi(z^k) = \begin{cases} (-1)^r \left(x_1^2 + x_2^2 + x_3^2\right)^l, & \text{if } k = 2l, \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

The case of $\operatorname{Re} \overline{z}^k$ is similar. □

Now we can finish the proof of Theorem 4.1.

Let $m \le n$ and note that

$$\operatorname{Re}\left(\overline{z}^m z^n\right) = |z|^{2m} \operatorname{Re} z^{n-m} = \begin{cases} (-1)^{\frac{n-m}{2}} \left(x_1^2 + x_2^2 + x_3^2\right)^{\frac{m+n}{2}}, & \text{if } n - m \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Next we calculate:

$$\begin{aligned} \int_{\mathbb{R}^3} \left(x_1^2 + x_2^2 + x_3^2\right)^l d\mu(z) &= \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{S}^2} \int_{\mathbb{R}} r^{2l+2} e^{-r^2/2} dr dS \\ &= \tfrac{2^{l+1}}{\sqrt{\pi}} \Gamma\left(l + \tfrac{3}{2}\right). \end{aligned}$$

Hence,

$$\int \overline{z}^m z^n d\mu(z) = (-1)^{\frac{n-m}{2}} \frac{2^{\frac{m+n}{2}+1}}{\sqrt{\pi}} \Gamma\left(\frac{m+n+3}{2}\right) = (-1)^{\frac{n-m}{2}} (m+n+1)!!$$

□

Appendix D: Proof of Theorem 5.5

We can express Q-polynomials in terms of Hermite polynomials.

$$Q_n(x) = \begin{cases} \frac{1}{s} H_{n+1}(s), & \text{if } n \text{ is even,} \\ \frac{1}{s^2}\left(s H_{n+1}(s) + H_n(s)\right), & \text{if } n \text{ is odd.} \end{cases} \tag{33}$$

Indeed, the first monic Hermite polynomials are $H_0 = 1$, $H_1 = x$, $H_2 = x^2 - 1$, and $H_3 = x^3 - 3x$, and a direct verification shows that the statement holds for $n = 0$ and $n = 1$, and that the polynomials on the right hand side of (33) satisfy the same 3-term recurrence as Q_n.

Then, by using the Plancherel–Rotach asymptotic formulas for the Hermite polynomials (see Appendix B and Szego [15, Theorem 8.22.9]), we can derive the asymptotic expressions for the Q-polynomials and for the functions $\rho_n(s,t)$ and $\delta_n(s,t)$.

For $\delta_n(s,t)$, this calculation leads to the conclusion that $\delta_n(s,t) = O(n^{-1})$. For $\rho_n(s,t)$, the situation is more complicated. First, we get

$$\begin{aligned} \rho_n(s,t)e^{-\frac{s^2+t^2}{4}} &= \sqrt{\frac{2}{\pi}}\frac{1}{(s-t)st}\Bigg[(\sin\varphi_{n+2}\sin\psi_{n+1})^{-1/2} \\ &\quad \times \sin\left(\left(\frac{n}{2}+\frac{5}{4}\right)\overline{\varphi_{n+2}|} + \frac{3\pi}{4}\right)\sin\left(\left(\frac{n}{2}+\frac{3}{4}\right)\overline{\psi_{n+1}|} + \frac{3\pi}{4}\right) \\ &\quad - (\sin\varphi_{n+1}\sin\psi_{n+2})^{-1/2} \\ &\quad \times \sin\left(\left(\frac{n}{2}+\frac{5}{4}\right)\overline{\psi_{n+2}|} + \frac{3\pi}{4}\right)\sin\left(\left(\frac{n}{2}+\frac{3}{4}\right)\overline{\varphi_{n+1}|} + \frac{3\pi}{4}\right)\Bigg] \\ &\quad + O\left(\frac{1}{\sqrt{n}}\right), \end{aligned} \tag{34}$$

where

$$\begin{aligned} \varphi_{n+1} &= \arccos\left(\frac{x}{2\sqrt{n+3/2}}\right), \\ \varphi_{n+2} &= \arccos\left(\frac{x}{2\sqrt{n+5/2}}\right), \end{aligned}$$

and similar expressions hold for ψ_{n+1} and ψ_{n+2}.

In order to get a bit simpler expression, we express φ_{n+2} and ψ_{n+2} in terms of φ_{n+1} and ψ_{n+1} and expand this expression in powers of n^{-1}. (For parsimony, we write φ_1, φ_2, ψ_1 and ψ_2 for φ_{n+1}, φ_{n+2}, ψ_{n+1} and ψ_{n+2}, respectively.)

$$\varphi_2 = \varphi_1 + \frac{\cos\varphi_1}{\sin\varphi_1}\left(\frac{1}{2n} - \frac{9}{8n^2}\right) - \frac{1}{8}\frac{\cos^3\varphi_1}{\sin^3\varphi_1}\frac{1}{n^2} + O\left(\frac{1}{n^3}\right)$$

A similar formula holds for ψ_2.

Then,

$$\begin{aligned} \overline{\varphi_2|} &= \overline{\varphi_1|} + 2[\cos(2\varphi_1) - 1]\frac{\cos\varphi_1}{\sin\varphi_1}\frac{1}{2n} + O\left(\frac{1}{n^2}\right) \\ &= \overline{\varphi_1|} - \frac{\sin(2\varphi_1)}{n} + O\left(\frac{1}{n^2}\right), \end{aligned} \tag{35}$$

and a similar formula holds for $\overline{\psi_2|}$.

By using the elementary trigonometric identities $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ and $\cos(x+3\pi/2) = \sin x$, we re-write several terms in (34):

$$\begin{aligned}
&\sin\left[\left(\frac{n}{2}+\frac{5}{4}\right)\overline{\varphi_2}+\frac{3\pi}{4}\right]\sin\left[\left(\frac{n}{2}+\frac{3}{4}\right)\overline{\psi_1}+\frac{3\pi}{4}\right]\\
&=\frac{1}{2}\left\{\cos\left[\left(\frac{n}{2}+\frac{3}{4}\right)(\overline{\varphi_2}-\overline{\psi_1})+\frac{1}{2}\overline{\varphi_2}\right]\right.\\
&\quad\left.-\sin\left[\left(\frac{n}{2}+\frac{3}{4}\right)(\overline{\varphi_2}+\overline{\psi_1})+\frac{1}{2}\overline{\varphi_2}\right]\right\},
\end{aligned}$$

and

$$\begin{aligned}
&\sin\left[\left(\frac{n}{2}+\frac{3}{4}\right)\overline{\varphi_1}+\frac{3\pi}{4}\right]\sin\left[\left(\frac{n}{2}+\frac{5}{4}\right)\overline{\psi_2}+\frac{3\pi}{4}\right]\\
&=\frac{1}{2}\left\{\cos\left[\left(\frac{n}{2}+\frac{3}{4}\right)(\overline{\varphi_1}-\overline{\psi_2})-\frac{1}{2}\overline{\psi_2}\right]\right.\\
&\quad\left.-\sin\left[\left(\frac{n}{2}+\frac{3}{4}\right)(\overline{\varphi_1}+\overline{\psi_2})+\frac{1}{2}\overline{\psi_2}\right]\right\}.
\end{aligned}$$

Next, note that both $(\sin\varphi_2\sin\psi_1)^{-1/2}$ and $(\sin\varphi_1\sin\psi_2)^{-1/2}$ in Formula (34) equal $(\sin\varphi_1\sin\psi_1)^{-1/2}+O(n^{-1})$. Hence, we can combine terms and use (35) to obtain

$$\begin{aligned}
&\cos\left(\frac{2n+3}{4}(\overline{\varphi_2}-\overline{\psi_1})+\frac{1}{2}\overline{\varphi_2}\right)-\cos\left(\frac{2n+3}{4}(\overline{\varphi_1}-\overline{\psi_2})-\frac{1}{2}\overline{\psi_2}\right)\\
&=\cos\left(\frac{2n+3}{4}(\overline{\varphi_1}-\overline{\psi_1})-\varphi_1\right)-\cos\left(\frac{2n+3}{4}(\overline{\varphi_1}-\overline{\psi_1})+\psi_1\right)+O\left(\frac{1}{n}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\sin\left(\frac{2n+3}{4}(\overline{\varphi_1}+\overline{\psi_2})+\frac{1}{2}\overline{\psi_2}\right)-\sin\left(\frac{2n+3}{4}(\overline{\varphi_2}+\overline{\psi_1})+\frac{1}{2}\overline{\varphi_2}\right)\\
&=\sin\left(\frac{2n+3}{4}(\overline{\varphi_1}+\overline{\psi_1})-\psi_1\right)-\sin\left(\frac{2n+3}{4}(\overline{\varphi_1}+\overline{\psi_1})-\varphi_1\right)+O\left(\frac{1}{n}\right).
\end{aligned}$$

With these modifications, Formula (34) implies the statement of the theorem. □

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